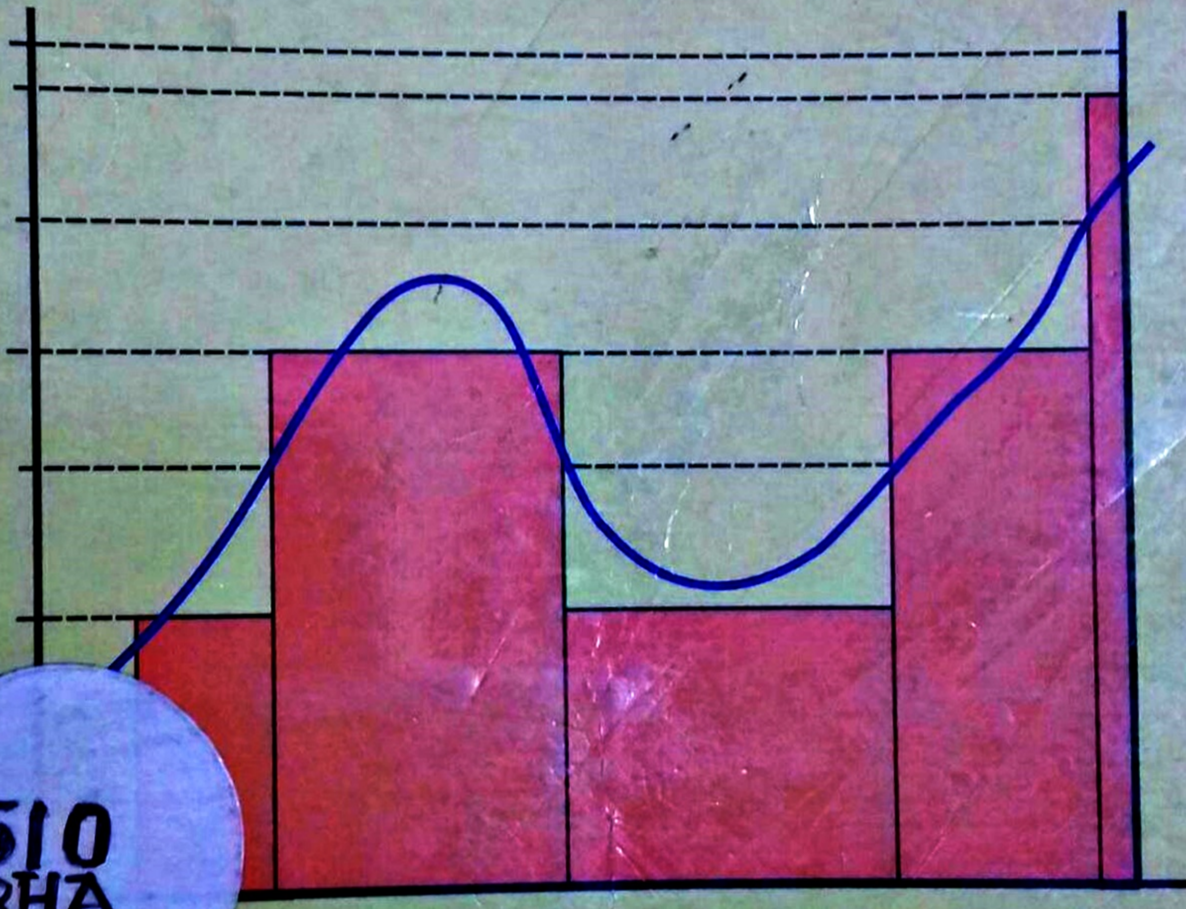
A COURSE OF
LEBESGUE INTEGRATION



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Bhakta A Course of Lebesgue Integration

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Lebesgue Integration

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Preface

The present book is written for the M.A. / M.Sc. Course of Mathematics in Indian Universities.

The book comprises ten chapters. The first chapter deals with, in short, the sets, set union and intersection, cartesian product of sets, mappings, countable sets, function on subsets of real line, covering by open intervals, Lindelof's Theorem, Heine-Borel Theorem, directed sets and nets, Weierstrass's approximation Theorem.

Second chapter deals with set rings, σ-rings and monotone classes.

In the third Chapter measures and outer measures are introduced ; measurable sets and extension of measures are given. Lebesgue and Lebesgue-stieltjes measures on the real line are discussed.

In Chapter four measurable furctions and their properties, convergence in measure have been discussed.

In Chapter five Lebesgue Integral of *a* bounded function on *a* measurable set with finite measure is introduced. Various properties of integrable functions have been discussed.

Chapters six and seven deal with the extension of Lebesgue Integral first to non-negative functions and then to general functions. Various important results have been given.

Chapter eight deals with functions of bounded variation and absolutely continuous functions and their differentiability.

In Chapter nine Fourier series has been discussed using Lebesgue Integral. Convergence Tests and other important results including Riesz-Fischer Theorem, Fejer's Theorem on $(c-1)$ summability of Fourier series have been discussed using Lebesgue Integral.

Last chapter deals with the density of sets, approximate continuity and differentiability.

The present author is indebted to those authors whose books have been consulted in writing the present book.

Author

CONTENTS

CHAPTER—I

SETS AND FUNCTIONS

Set and union and intersection of sets.

A set is *a* callection of obijects according to some rule ; objects are called elements. Sets are generally denoted by block capital letters $A, B, C, \ldots$ and the elements by small letters $a, b, c, \ldots$ with or without subscripts. If an element x belongs to set A we write $x \in A$ and if x does not belong to A we write $x \notin A$. It is convenient to have *a* set containing no element ; we call this set the void set or empty set and denote it by ϕ.

Two sets A and B are said to be equal if they consist of exactly same elements and we write $A = B$ and its negation by $A \neq B$. We say that *a* set A is *a* subset of the set B or A is contained in the set B if each element of A is an element of B. and we write $A \subset B$. If $A \subset B$ and $A \neq B$, then A is called *a* proper subset of B. It is obvious that $\phi \subset A$ for every set A.

Let us confine our consideration to the elements of *a* nonempty set X and call the set X as universal set. Let $\mathscr{F} = \{A_\alpha : \alpha \in \Delta\}$ be *a* class of subsets of X. We form *a* set S such that an element $x \in S$ if $x \in A_\alpha$ for some α in Δ and call S the union of the sets $A_\alpha \in \mathscr{F}$ and write

$S = \cup \{A_\alpha : \alpha \in \Delta\}$ or $S = \cup_{\alpha \in \Delta} A_\alpha$ or $S = \cup_\alpha A_\alpha$.

Next, we form another set M such that an element $x \in M$ if $x \in A_\alpha$ for every α in Δ. We call M the intersection of the sets $A_\alpha \in \mathscr{F}$ and write

$M = \cap \{A_\alpha : \alpha \in \Delta\}$ or $M = \cap_{\alpha \in \Delta} A_\alpha$.

If $\Delta = \{1, 2, 3, \ldots, n\}$, then the union and the intersection of the sets $A_1, A_2, \ldots, A_n$ are written as

$$S = \bigcup_{i=1}^{n} A_i \text{ and } M = \bigcap_{i=1}^{n} A_i.$$

If $A \cap B = \phi$, we say that the sets A and B are disjoint.

Let A and B be two sets. The set of all elements of A which are not in B is denoted by A/B. The set X/A is called the complement of A (*w,.r, to* X) and is denoted by A'.

Theorem 1.1. (*D'* Morgan's Theorem).

Let $\mathscr{F} = \{A_\alpha : \alpha \in \Delta\}$ be *a* nonempty class of sets. Then
(i) $(\cup_\alpha A_\alpha)' = \cap_\alpha A'_\alpha$. (ii) $(\cap_\alpha A_\alpha)' = \cup_\alpha A'_\alpha$

Proof : We prove the relation (i). The proof of the other is analogous. Write $A = (\cup_\alpha A_\alpha)'$ and $B = \cap_\alpha A'_\alpha$.

Let $x \in A$. Then $x \notin \cup_\alpha A_\alpha$ which gives that $x \notin A_\alpha$ for any α and so $x \in A'_\alpha$ for all α. This implies that $x \in \cap_\alpha A'_\alpha = B$. Thus

$$A \subset B. \quad \ldots \quad \ldots \quad (1)$$

Next, let $x \in B$. Then $x \in A'_\alpha$ for all α which gives that $x \notin A_\alpha$ for any α and so $x \notin \cup_\alpha A_a$. This implies that $x \in (\cup_\alpha A_\alpha)' = A$. So

$$B \subset A. \quad \ldots \quad \ldots \quad (2)$$

From (1) and (2) weget A = B.

Cartesian product of sets.

Let A and B be two nonempty sets. The set of all pairs (x, y) $(x \in A, y \in B)$ is called the Cartesian product of the sets A and B and is denoted by $A \times B$. Thus

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}.$$

If $A = \phi$ or $B = \phi$ we define $A \times B = \phi$. If we have n sets $A_1, A_2, \ldots, A_n$, *then the set of all* n-tuples $(x_1, x_2, -, x_n)$ with $x_i \in A_i$ is called the Cartesian product of the sets $A_1, A_2, \ldots, A_n$ and is denoted by

$$\pi\{A_i : i = 1 ; 2, \ldots, n\} \text{ or } \pi_{i=1}^{n} A_i.$$

Mapping or function.

Let A and B be two non-empty sets. *A mapping or function* f from A to B is *a* rule which assigns to each element x of A *a* definite element y in B; y is called the image of x under the mapping f and is written as $y = f(x)$. A is called the domain of f and B is called the co-domain of f and is expressed as $f : A \to B$.

Let E be *a* subset of A and F be *a* subset of B. We define $f(E) = \{y : y = f(x)$ for some $x \in E\}$ and $f^{-1}(F) = \{x : x \in A$ and $f(x) \in F\}$. The set $f(A)$ is called the range of f.

(I) A mapping $f : A \to B$ is said to be injective or one-to-one if for x_1, x_2 in A and $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$.

(II) A mapping $f : A \to B$ is said to be surjective or onto if $f(A) = B$.

(III) A mapping $f : A \to B$ is said to be bijective if it is both injective and surjective, that is, one—to—one and onto.

(IV) Let $f : A \to B$ and $g : A \to B$. If $f(x) = g(x)$ for all x in A we say that f is equal to g and write $f = g$.

(V) A mapping $f : A \to A$ is said to be identity mapping on A if $f(x) = x$ for all x in A; we write it as $f = I_A$.

Composite mapping.

Let $f : A \to B$ and $g : C \to D$ be two mappings such that $f(A) \subset C$. Let

$x \in A$. Then $f(x) \in f(A)$. Write $y = f(x)$. Then $y \in C$ and so $g(y) \in D$. Let us define $h : A \to D$ as follows :

$h(x) = g(f(x))$ for $x \in A$.

This mapping h is called the composite mapping of f and g (or product of f and g) and is denoted by gof or gf. Thus (gof) $(x) = g(f(x))$ or

$$(gf)\,(x) = g(f(x)) \text{ for } x \in A.$$

Theorem 1.2. Let $f : A \to B$, $g : B \to C$ and $h : C \to D$ be three mappings. Then

$$h(gf) = (hg)f.$$

Proof : Write $p = h\,(gf)$ and $q = (hg)f$. Take any $x \in A$. Then $y = f(x) \in B$, $z = g(y) \in C$ and $w = h(z) \in D$. We have

$p(x) = [h(gf)]\,(x) = h((gf)(x)) = h\,(g(f(x))) = h(g(y)) = h(z) = w.$

$q(x) = [(hg)f]\,(x) = (hg)(f(x)) = (hg)(y) = h(g(y)) = h(z) = w.$

Thus $p(x) = q(x)$ for all $x \in A$.

So $p = q$, that is, $h(gf) = (hg)f$.

Theorem 1.3. Let the mappings $f : A \to B$ and $g : B \to C$ be both injective. Then the composite mapping gf is injective.

Proof : Let x_1, x_2 $(x_1 \neq x_2)$ be any two points in A. Write $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since f is injective, $f(x_1) \neq f(x_2)$, that is, $y_1 \neq y_2$. Again, g is injective and so $g(y_1) \neq g(y_2)$. We have

$(gf)\,(x_1) = g\,(f(x_1)) = g(y_1)$ and $(gf)\,(x_2) = g(f(x_2)) = g(y_2)$.

Thus $(gf)\,(x_1) \neq (gf)\,(x_2)$. So gf is injective.

Theorem 1.4. Let $f : A \to B$ and $g : B \to C$ be two mappings such that the composite, mapping gf is injective. Then f is injective.

Proof : Let x_1, x_2 $(x_1 \neq x_2)$ be any two elments in A. Write $y_1 = f(x_1)$ and $y_2 = f(x_2)$. We have

$(gf)\,(x_1) = g\,(f(x_1)) = g(y_1)$ and $(gf)\,(x_2) = g(f(x_2)) = g(y_2)$.

Since gf is injective and $x_1 \neq x_2$, we get $g(y_1) \neq g(y_2)$ which implies that $y_1 \neq y_2$, that is, $f(x_1) \neq f(x_2)$.

Hence f is injective.

Inverse mapping.

A mapping $f : A \to B$ is said to be invertibe (or said to have an inverse)if there is a mapping $g : B \to A$ such that gof $= I_A$. and fog $= I_B$ Then g is called an inverse of f and we write $g = f^{-1}$.

Let g and h be two inverses of the mapping $f : A \to B$. Then $gf = I_A$, $hf = I_A$ and $fg = I_B$, $fh = I_B$. We have

$g = gI_B = g(fh) = (gf)\,h = I_A h = h.$

Hence f has unique inverse when it has an inverse.

Theorem 1.5. *A mapping $f : A \to B$ is invertible if and only if f is a bijection.*

Proof : Let $f : A \to B$ be a bijection. Then f is one-to-one and onto. Let $y \in B$. Then there is only one element x in A such that $y = f(x)$. Define $g : B \to A$ by $g(y) = x$. We have $(fg)(y) = f(g(y)) = f(x) = y$. This gives that $fg = I_B$.

Again let $x \in A$ and let $y = f(x)$. Then $y \in B$. By definition $g(y) = x$. We have $(gf)(x) = g(f(x)) = g(y) = x$.

So $gf = I_A$. Hence g is the inverse of f, that is, f is invertible.

Next, let $f : A \to B$ be invertible. Then there is a mapping $g : B \to A$ such that $gf = I_A$ and $fg = I_B$.

Let x_1 and x_2 $(x_1 \neq x_2)$ be any elements in A. We have

$x_1 = I_A(x_1) = (gf)(x_1) = g(f(x_1))$.

and $x_2 = I_A(x_2) = (gf)(x_2) = g(f(x_2))$.

Since $x_1 \neq x_2$, $g(f(x_1)) \neq g(f(x_2))$. This implies that $f(x_1) \neq f(x_2)$. Hence f is injective,

Now let $y \in B$, Then $x = g(y) \in A$ and $y = I_B(y) = (fg)(y) = f(g(y) = f(x)$. This gives that f is onto. Hence f is a bijection.

Equinumerous sets.

Two sets A and B are said to be equinumerous if there is a bijection f of A to B and we write $A \sim B$. If f is a bifection of A to B, then f^{-1} is a bijection of B to A. So $B \sim A$.

Let $A \sim B$ and $B \sim C$, Then there are bijections $f : A \to B$ and $g : B \to C$. Clearly gf is a bijection of A to C. So $A \sim C$. Since I_A is a bijection of A to A, $A \sim A$.

A set A is said to be finite if A is equinumerous to the set $E = \{1,2,3,..., n\}$ for some positive integer n. Void set ϕ is considered as a finite set.

A set E is said to be infinite if it is not finite.

Countable sets.

A set E is said to be countableif it is finite or equinvmerous to the set N of all natural numbers.

Theorem 1.6. *A nonempty set E is countable if and only if its elements can be taken as*

$a_1, a_2, a_3, \ldots\ldots, a_n, \ldots\ldots$ where $a_i \neq a_j$ for $i \neq j$.

Proof : First suppose that the set E is countable. If E is finite, then it is equinumerous to the set $A = \{1,2,3,...., n\}$ for some positive integer n. Then

there is a bijection f of A to E. Clearly $f(1), f(2), f(3), \ldots, f(n)$ are elements of E and E has no other element.

Write $a_i = f(i)$ $(i = 1,2,3, \ldots, n)$

Since $f(j) \neq f(i)$ for $i \neq j$ we have $a_i \neq a_j$ for $i \neq j$. Thus the elements of E are

$a_1, a_2, a_3, \ldots, a_n$ where $a_i \neq a_j$ for $i \neq j$.

Next, suppose that the set E is infinite. Then E is equinumerous to the set N of all natural numbers. Then there is a bijection f of N to E. Clearly $f(i) \in E$ for all $i \in N$ and E has no other element. Write $a_i = f(i)$ $(i = 1,2,3, \ldots, n, \ldots)$

Since f is a bijection $f(i) \neq f(j)$ for $i \neq j$, that is, $a_i \neq a_j$ for $i \neq j$.

Thus the elements of E are $a_1, a_2, a_3, \ldots, a_n, \ldots$

where $a_i \neq a_j$ for $i \neq j$.

Lastly, suppose that the elements of E can be taken as

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

where $a_i \neq a_j$ for $i \neq j$.

If E is finite, then clearly E is countable. Let E be infinite. Define

$f : E \to N$ as follows.

$$f(a_i) = i \ (i = 1, 2, 3, \ldots).$$

Then clearly f is a bijection of E to N. So E is equinumerous to the set N and hence E is countable.

Theorem 1. 7. If the set E is countable, then every subset of E is countable.

Proof : Let A be a subset of E. If A is finite, then A is countable. Suppose that the set A is infinite. Then E is also infinite. Since E is countable the elements of E can be taken as

$$a_1, a_2, a_3, \ldots, a_n, \ldots \text{where } a_i \neq a_j \text{ for } i \neq j. \quad (1)$$

If we proceed along the row (1) starting from a_1 we shall meet the elements of A time to time. Denote by b_1 the element of A which we meet first, by b_2 the element of A which we meet second and so on. Thus the elements of A can be taken as

$b_1, b_2, b_3, \ldots, b_n, \ldots$

Clearly $b_i \neq b_j$ for $i \neq j$. Hence the set A is countable.

Theorem 1.8. The set $N \times N$ is countable, where N is the set of all positive integers.

Proof : Let $E = N \times N$. An element of E is an ordered pair (m,n), where m and n are positive integers. We define the mapping $f : E \to N$ as follows.

$f(m,n) = 2^m 3^n$ for $(m, n) \in E$.

Let $A = f(E)$. Then A is a subset of N and so A is countable.

Let (m,n) and (p,q) be two elements of E and let $(m, n) \neq (p, q)$. Then

clearly $2^m 3^n \neq 2^p 3^q$ which gives that $f(m,n) \neq f(p,q)$ So the mapping f is one-to-one. Therefore E is countable.

Corollary 1.8.1. If the sets A and B are countable, then the set $A \times B$ is countable.

Corollary 1.8.2. If the sets $A_1, A_2, ..., A_n$ are countable then the set $A_1 \times A_2 \times A_3 \times A_n$ is countable.

Theorem 1.9. If $\{A_i\}_{i=1}^{\infty}$ is *a* sequence of countable sets, then the set $A = \bigcup_{i=1}^{\infty} A_i$ is countable.

Proof : (I) We first suppose that the sets $A_1, A_2, A_3, \ldots, A_n, \ldots$ are pairwise disjoint. Since the set A_i countable its elements can be taken as

$A_i = \{a_{i1}, a_{i2}, a_{i3}, \ldots\ldots\ldots\}$ where $a_{ij} \neq a_{ik}$ for $j \neq k$.

Since the sets $A_1, A_2, A_3, \ldots$ are pairwise disjoint, $a_{ij} \neq a_{rs}$ when $(i,j) \neq (r,s)$. We define the mapping $f : A \rightarrow N \times N$ as follows.

$f(a_{mn}) = (m,n)$ for $a_{mn} \in A$.

Let $B = f(A)$. Then B is *a* subset of $N \times N$. So B is countable.

Let a_{mn} and a_{rs} be two elements of A and let $a_{mn} \neq a_{rs}$. Then clearly $(m,n) \neq (r, s)$ and so $f(a_{mn}) \neq f(a_{rs})$. This gives that the mapping f is one–to–one. Hence the set A is countable.

(II) Next, suppose that the sets $A_1, A_2, A_3, \ldots$ are not pairwise disjoint. We define the sets $B_1, B_2, B_3, \ldots\ldots$ as follows.

$B_1 = A_1, B_2 = A_2 / A_1, B_3 = A_3 / (A_1 \cup A_2),$

$B_4 = A_4 / (A_1 \cup A_2 \cup A_3), \ldots\ldots\ldots$

Then the sets $B_1, B_2, B_3, \ldots\ldots$ are pairwise disjoint and $A = \bigcup_{i=1}^{\infty} B_i$. Since $B_i \subset A_i$, B_i is countable. Hence by case *I*, the set A is countable.

Corollary 1.9.1. If the sets $A_1, A_2, ..., A_n$ are countable, then the set $A = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$ is countable.

Corollary 1.9.2. The set of all rational numbers is countable.

Proof : For each positive integer $n.$, let

$$A_n = \left\{\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{m}{n}, \ldots\right\}.$$

Denote by Q_+ the set of all positive rational numbers, by Q_- the set of all negative rational numbers and by Q the set of all rational numbers.

Then $Q_+ = \bigcup_{m=1}^{\infty} A_n$ and $Q = Q_+ \cup Q_- \cup \{o\}$. Since each A_n is countable, Q_+ is countable. Also $Q_- = \{(-x) ; x \in Q_+\}$ which gives that Q_- is countable. Hence Q is countable.

Nest of intervals.

A sequence $\{I_n\}$ of intervals is said to form *a* nest if the following conditions hold.

(i) $I_{n+1} \subset I_n$ $(n = 1, 2, 3, ...)$

(ii) $|I_n| \to 0$ *as* $n \to \infty$,

where $|I_n|$ denotes the length of the interval I_n.

Theorem 1.10. (Theorem on nested intervals).

If the sequence $\{I_n\}$ of closed intervals forms *a* nest, then there exists only one real number ξ such that $\xi \in I_n$ for all n.

Proof : Let $I_n = [a_n, b_n]$ $(n = 1, 2, 3, ...)$. Since the sequence $\{I_n\}$ forms *a* nest, we have

$[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ $(n = 1, 2, 3, ...)$. This gives that

$a_1 \le a_2 \le a_3 \le a_n < b_n \le ... \le b_3 \le b_3 \le b_2 \le b_1$.

The sequence $\{a_n\}$ is increasing and the sequence $\{b_n\}$ is decreasing and each of them is bounded. So each of them is convergent.

Let $\xi = \lim a_n$ and $\eta = \lim b_n$.

Clearly $a_n \le \xi \le b_n$ and $a_n \le \eta \le b_n$ for all n.

Since $|I_n| = b_n - a_n \to 0$ as $n \to \infty$ we have $\xi = \eta$. Then $a_n \le \xi \le b_n$ for all n.

Let ξ' be any real number such that $\xi' \in I_n$ for all n. Then $a_n \le \xi' \le b_n$ for all n. So $1\xi' - \xi| \le b_n - a_n$ for all n. This gives that $\xi' = \xi$.

Hence there is only one real number ξ such that $\xi \in I_n$ for all n.

Theorem 1.11. Let $A = \{x : 0 \le x \le 1\}$. Then the set A is uncountable.

Proof : Assume that the set A countable. Then its elements can be taken as

(1) $x_1, x_2, ..., x_n, ...$ where $x_i \neq x_i$ for $i \neq j$

We divide the interval $[0,1]$ into three closed intervals

$\left[0, \frac{1}{3}\right], \left[\frac{1}{3}, \frac{2}{3}\right], \left[\frac{2}{3}, 1\right]$ of equal length. Clearly the point x_1 does not belong to at least one of them. We denote such an interval by I_1. Next we divide the interval I_1 into three closed intervals I_1', I_1'', I_1''' of equal length. We denote by I_2 one of these intervals which does not contain the point x_2. Proceeding in this way we obtain *a* sequence $\{I_n\}$ of closed intervals with the following properties.

(i) $I_{n+1} \subset I_n$ (ii) $|I_n| = \dfrac{1}{3^n}$ and (iii) $x_n \notin I_n$ $(n = 1, 2, 3, ...)$.

From (i) and (ii) we see that the sequence $\{I_n\}$ forms *a* nest. So there is only one real number ξ such that $\xi \in I_n$ for all *n*. Since $I_n \subset [0, 1]$, $\xi \in [0,1]$. So $\xi = x_n$ for some positive integer *n* because all the elements of [0,1] occur in the row (1). This contradicts (iii). Hence the set *A* is uncountable.

Corollary 1.11.1. The set *R* of real numbers is uncountable.

Corollary 1.11.2. Let *a* and be $(a < b)$ be any two real numbers, and let $A = \{x : a \leq x \leq b\}$. Then *A* is uncountable.

Let $B = [0, 1]$. Consider the mapping $f : A \to B$. defined by

$$f(x) = \frac{x-a}{b-a} \text{ for } x \in A.$$

Then *f* is *a* bijection of *A* to *B*. Since *B* is uncountable, *A* is uncountable.

Open and closed sets on the real line.

We denote by *R* the set of all real numbers. Let *E* be *a* subset of *R*. A point x_0 in *R* is said to be an interior point of the set *E* if there *a* $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset E$. The set *E* is said to be open if every point of *E* is an interior point of *E*.

Let $x_0 \in R$. A subset *U* of *R* is said to be *a* neighbourhood (in short neighd) of the point x_0 if $(x_0 - \delta, x_0 + \delta) \subset U$ for some $\delta > 0$, that is, x_0 is an interior point of x_0. Clearly for $\delta > 0$, $(x_0 - \delta, x_0 + \delta)$ is *a* neighd of the point x_0.

Let $x_0 \in R$ and *E* be *a* subset of *R*.The point x_0 is said to be *a* limit point of the set *E* if $(E|\{x_0\}) \cap U \neq \phi$ for every neighd *U* of x_0. The set *E* is said to be closed if it contains all its limit points.

From definition we see that the set ϕ and the set *R* are open as well as closed.

Theorem 1.12. *A* subset *E* of *R* is closed if and only if its complement E' is open.

Proof : First suppose that the set *E* is closed. Then it contains all its limit points. Let x_0 be any point of E'. Then x_0 is not *a* limit point of the set *E*. So there is *a* neighd *U* of x_0 such that $(E|\{x_0\}) \cap U = \phi$. This gives that $U \subset E'$. So x_0 is an interior point of E'. Since x_0 is arbitrary, it follows that E' is open.

Next, let E' be open. Take any point $x_0 \in E'$. Then $(x_0 - \delta, x_0 + \delta) \subset E'$ for some $\delta > 0$ which gives that $(x_0 - \delta, x_0 + \delta)$ does not contain any point of *E*, So x_0 is not *a* limit point of *E*.

Thus no point of E' is *a* limit point of *E*. This implies that *E* contains all its limit points. Hence the set *E* is closed.

Theorem 1.13. The intersection of finite number of open sets is open and the union of any non-empty family of open sets is open.

Proof : Let $G_1, G_2, \ldots, G_n$ be open sets and $G = \bigcap_{i=1}^{n} G_i$. If $G = \phi$, then G is open. Suppose that $G \neq \phi$. Let $x_0 \in G$. Then $x_0 \in G_i$ $(i=1, 2, \ldots, n)$. So there are positive numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $(x_0 - \delta_i, x_0 - \delta_i) \subset G_i$ Let $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_n\}$. Then $\delta > 0$ and $(x_0 - \delta, x_0 + \delta) \subset (x_0 - \delta_i, x_0 + \delta_i) \subset G_i$ for $i = 1, 2, \ldots, n$. So $(x_0 - \delta, x_0 + \delta) \subset \bigcap_{i=1}^{n} G_i = G$. Thus x_0 is an interior point of G. Since x_0 is arbitrary, it follows that G is open.

Next, let $\mathscr{F} = \{G_\alpha : \alpha \in \Delta\}$ be *a* nonempty family of open sets and let $G = \cup\{G_\alpha : \alpha \in \Delta\}$. Take any point $x_0 \in G$. Then $x_0 \in G_{\alpha_0}$ for some $\alpha_0 \in \Delta$. Since G_{α_0} is open $(x_0 - x_0 + \delta) \subset G_{\alpha_0}$ for some $\alpha_0 \in \Delta$. Since $G_{\alpha_0} \subset G$, we have $(x_0 - \delta, x_0 + \delta) \subset G$. So x_0 is an interior point of G. Since x_0 is abitrary, G is open.

Theorem 1.14. The union of finite number of closed sets is closed and the intersection of any nonempty family of closed sets is closed.

Proof : Let $F_1, F_2, \ldots, F_n$ be closed sets and $F = \bigcup_{i=1}^{n} F_i$. Let x_0 be *a* limit point of F. Assume that $x_0 \notin F$. Then $x_0 \notin F_i$ $(i = 1, 2, \ldots, n)$. This implies that there are positive numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $(x_0 - \delta_i, x_0 + \delta_i) \cap F_i = \phi$ $(i = 1, 2, \ldots, n)$. Let $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_n\}$. Then $(x_0 - \delta, x_0 + \delta) \cap F_i = \phi$ $(i = 1, 2, \ldots, n)$ and so $(x_0 - \delta, x_0 + \delta) \cap F = \phi$, This contradicts the fact that x_0 is *a* limit point of F. So $x_0 \in F$ and this gives that F is closed.

Next, let $\mathscr{F} = \{F_\alpha : \alpha \in \Delta\}$ be any nonempty family of closed sets and let $F = \cap\{F_\alpha : \alpha \in \Delta\}$.

Let x_0 be any limit point of F. Then $(x_0 - \delta, x_0 + \delta) \cap (F \,|\, \{x_0\}) \neq \phi$ for every $\delta > 0$. Since $F \subset F_\alpha$ we have $(x_0 - \delta, x_0 + \delta) \cap (F_\alpha | \{x_0\}) \neq \phi$ for every $\delta > 0$. So x_0 is *a* limit point of F_α. Since F_α is closed, $x_0 \in F_\alpha$. So $x_0 \in \cap\{F_\alpha : \alpha \in \Delta\}$. Hence F is closed.

Note 1.1. (1) The intersecton of infinite number of open sets may not be open.

$$\text{Let } G_n = \left(-\frac{1}{n}, 1+\frac{1}{n}\right) \quad (n = 1,2,3, \ldots).$$

For each n, G_n is open. We have $\bigcap_{n=1}^{\infty} G_n = [0, 1]$ which is closed.

(2) The union of infinite number of closed sets may not be closed.

Let $F_n = \left[\frac{1}{n}, 1\right]$ $(n = 1, 2, 3, \ldots)$

For each n, F_n is closed. We have $\bigcup_{n=1}^{\infty} F_n = (0, 1]$ which is not closed.

Theorem 1.15. Every nonempty bounded open set on the real line can be expressed as the union of a countable family of pairwise disjoint open intervals.

Proof : Let G be a nonempty bounded open set on the real line. Take any point x in G. Then x is an interior point of G. So $(x-\delta, x+\delta) \subset G$ for some $\delta > 0$.

Let $A_x = \{y : (y, x) \subset G\}$ and $B_x = \{z : (x, z) \subset G\}$.

Then A_x and B_x are non empty bounded sets of real numbers. Denote by a_x the glb of A_x and by b_x the lub of B_x. Let $I_x = (a_x, b_x)$. Then I_x is an open interval containing the point x. Take any $u \in I_x$ Then $a_x < u < b_x$. Let $x < u < b_x$. Then there is point z in B_x such that $u < z$. Then $(x, z) \subset G$ which gives that $u \in G$. If $a_x < u < x$ we can show that $u \in G$. Thus $I_x \subset G$. We now show that none of a_x and b_x belongs to G. If possible, let $b_x \in G$. Then there is an $\eta > 0$ such that $(b_x, b_x + \eta) \subset G$ which gives that $b_x + \eta \in B_x$. This contradicts the fact that b_x is lub of B_x. Hence $b_x \notin G$, Similarly we can show that $a_x \notin G$. Thus to each x in G we get an open interval $I_x = (a_x, b_x)$ containing the point x such that $I_x \subset G$ but none of a_x and b_x belongs to G. Such an interval I_x is called a component interval of G.

Let $\mathcal{F} = \{I_x : x \in G\}$. Then clearly $G = \cup\{I_x : I_x \in \mathcal{F}\}$. Now we show that the family $\mathcal{F}$ is countable and the intervals of $\mathcal{F}$ are pairwise disjoint. Let (a,b) and (c,d) be any two intervals of the family $\mathcal{F}$. If possible, let $(a, b) \cap (c,d) \neq \phi$. Take any $\alpha \in (a, b) \cap (c, d)$. Then $a < \alpha < b$ and $c < \alpha < d$. This gives that $a < d$ and $c < b$. If $d < b$, then $a < d < b$ which gives that $d \in G$. Again, if $b < d$, $c < b < d$ which also gives that $b \in G$. Thus in any case we get a cantradiction because none of b and d belongs to G. Hence $b = d$. Similarly we get $a = c$. Thus two intervals of $\mathcal{F}$ are either identical or disjoint. Hence $\mathcal{F}$ is a family of pairwise disjoint open intervals. For each open interval I in $\mathcal{F}$ choose a rational number $\gamma_I \in I$ and let $E = \{\gamma_I : I \in \mathcal{F}\}$. Clearly E is countable. Define the mapping $f : \mathcal{F} \to E$ as follows.

$f(I) = r_I$ for $I \in \mathcal{F}$.

Let $I, J \in \mathcal{F}$ and $I \neq J$. Then $I \cap J = \phi$ which gives that $\gamma_I \neq \gamma_J$. So $f(I) \neq f(J)$ for $I \neq J$. Therefore the mapping f is one-to-one. Since E is countable, the family $\mathcal{F}$ is countable.

Note 1.2. If G is an unbounded open set on the real line, then the result is also true.

Covering by open intervals.

Let E be a subset of R and $\mathscr{F}$ be a family of open intervals. We say that the family $\mathscr{F}$ covers the set E if

$E \subset \cup \{I : I \in \mathscr{F}\}]$.

Let $\mathscr{F}$ and $\mathscr{F}^*$ be two covers of the set E. If $\mathscr{F}^* \subset \mathscr{F}$, then $\mathscr{F}^*$ is called a subcover of the set E. Further if $\mathscr{F}^*$ consists of only finite number of intervals, then $\mathscr{F}^*$ is called a finite subcover of E.

Theorem 1.16. (Lindelof Theorem).

Let E be a subset of Rand $\mathscr{F}$ be a family of open intervals which covers the set E. Then there is a countable subfamily $\mathscr{F}^*$ of $\mathscr{F}$ such that $\mathscr{F}^*$ also covers the set E.

Proof : Let E be a subset of R and $\mathscr{F}$ be a family of open intervals which covers the set E. Take any x in E Then there is an open interval I_x in $\mathscr{F}$ such that $x \in I_x$. Let $I_x = (a_x, b_x)$. Choose rational numbers α_x and β_x with $a_\alpha < \alpha_x < x < \beta_x < b_x$. Write $J_x = (\alpha_x, \beta_x)$. Then $x \in J_x$ and $J_x \subset I_x$. Let $\Delta = \{J_x : x \in E\}$. Clearly Δ is a cover of E. Since the set Q of all rational numbers is countable, it follows that Δ is countable. So its elements can be taken as

$J_1, J_2, J_3, \ldots, J_n, \ldots\ldots$

For each positive integer n, denote by I_n the open interval in $\mathscr{F}$ such that $J_n \subset I_n$. Let $\mathscr{F}^* = \{I_n : n = 1, 2, 3, \ldots.\}$. Then $\mathscr{F}^* \subset \mathscr{F}$ and $\mathscr{F}^*$ is countable.

Let $x \in E$. Then $x \in J_n$ for some positive integer n. Since $J_n \subset I_n$, $x \in I_n$. Hence $\mathscr{F}^*$ is a cover of E. This completes the proof of the theorem.

Diameter of a set.

Let E be a nonempty subset of R and let diam $(E) = \sup \{|x-y| : x, y \in E\}$. Then diam (E) is called the diameter of the set E.

Eample 1.1.

Let F be a nonempty bounded and closed subset of R amd let $\lambda = \inf F$ and $\mu = \sup F$. Then $\lambda, \mu \in F$.

Soln. Since F is bounded, there are real numbers a and b with $a < x < b$ for all x in F.

This gives that $a \leq \lambda \leq \mu \leq b$.

Assume that $\lambda \notin F$. Then $\lambda < x$ for all x in F. Choose any $\varepsilon > 0$. Then there is a point y in F such that $y < \lambda + \varepsilon$. Since $\lambda < y$ we see that $y \in (\lambda-\varepsilon, \lambda +$

ε) but $y \neq \lambda$. Since this is true for every $\varepsilon > 0$, it follows that λ is a limit point of F. So $\lambda \in F$ which contradicts our assumption. Hence $\lambda \in F$. Similarly we can show that $\mu \in F$.

Example 1.2.

Let F be a closed subset of R. If $\{x_n\} \subset F$ and $\{x_n\}$ converges to α, then $\alpha \in F$.

Soln. Assume that $\alpha \notin F$. Then $\alpha \in F'$, where F' is the complement of the set F. Since F is closed, F' is open. So we can find a $\delta > 0$ such that $(\alpha - \delta, \alpha + \delta) \subset F'$. Since $\{x_n\}$ converges to α, there is a positive integer n_0 such that

$x_n \in (\alpha - \delta, \alpha + \delta)$ for all $n \geq n_0$ that is, $x_n \in F'$ for all $n \geq n_0$.
This contradicts the fact that $x_n \in F$ for all n. Hence $\alpha \in F$.

Theorem 1.17. (Cantor's Intersection Theorem).

Let $\{F_n\}$ be a sequence of nonempty bounded closed sets such that $F_1 \supset F_2 \supset F_3 \supset \ldots\ldots \supset F_n \supset \ldots$

Then $\bigcap_{n=1}^{\infty} F_n$ is nonempty.

Further if diam $(F_n) \to 0$ as $n \to \infty$, then $\bigcap_{n=1}^{\infty} F_n$ consists of only one point.

Proof : Since F_1 is bounded there are real numbers a and b such that

$a < x < b$ for all $x \in F_1$. ... ... (1.1)

For each positive integer n, let $x_n = \sup F_n$. Then $x_n \in F_n$ for all n. Since $F_{n+1} \subset F_n$, it follows that $x_{n+1} \leq x_n$; also by (1.1) $a < x_n < b$ for all n. Thus the sequence $\{x_n\}$ is decreasing and bounded. So it is convergent. Let $\alpha = \lim x_n$.

Take any positive integer p. For any $n \geq p$, $F_n \subset F_p$. So $x_n \in F_p$ for $n \geq p$. Since F_p is closed, $\alpha \in F_p$.

This is true for every positive integer p. Hence $\alpha \in \bigcap_{p=1}^{\infty} F_p$.

Next, suppose that diam $(F_n) \to 0$ as $n \to \infty$. Let $\xi \in \bigcap_{p=1}^{\infty} F_p$. Then $\xi, \alpha \in F_p$ for all n. So

$|\xi - \alpha| \leq$ diam $(F_p) \to 0$ as $p \to 0$ as $p \to \infty$.

Hence $\xi = \alpha$. Therefore $\bigcap_{p=1}^{\infty} F_p$ consists of only one point.

Compact sets.

A subset E of the real line R is said to be compact if every family of open intervals covering the set E has a finite subcover.

Theorem 1.18. (Heine–Borel Theorem).

Let F be a nonempty bounded and closed set and let $\mathscr{F}$ be a family of open intervals which covers the set F. Then it is possible to choose finite number of open intervals $I_1, I_2, \ldots, I_n$ from the family $\mathscr{F}$ such that

$$F \subset I_1 \cup I_2 \cup \ldots \cup I_n.$$

Proof. By Lindelof's Theorem the family $\mathscr{F}$ has a countable subfamily $\mathscr{F}^* = \{I_1, I_2, I_3, \ldots\}$ which also covers the set F. If $\mathscr{F}^*$ is finite, there is nothing to prove. Suppose that $\mathscr{F}^*$ is infinite.

We define the sets $F_1, F_2, F_3, \ldots$ asfollows.

$F_1 = F \mid I_1$, $F_2 = F \mid (I_1 \cup I_2)$, $F_3 = F \mid (I_1 \cup I_2 \cup I_3)$, ...

$F_n = F \mid (I_1 \cup I_2 \cup \ldots\ldots \cup I_n)$

....

For each n we have

$F_n = F \cap I_1' \cap I_2' \cap \ldots. \cap I_n'$

where the prime ($'$) denotes the complement. Clearly each F_n is bounded and closed and

$F_1 \supset F_2 \supset F_3 \supset \ldots\ldots$

Assume that $F_k \neq \phi$ for all k. By Cantor's intersection Theorem the set $\bigcap_{n=1}^{\infty} F_n$ is nonempty.

Let $\alpha \in \bigcap_{n=1}^{\infty} F_n$. Then $\alpha \in F_n$ for all n. This gives that $\alpha \notin I_n$ for all n. which contradicts the fact that $\mathscr{F}^*$ covers the set F. Hence $F_m = \phi$ for some positive integer m. This gives that

$$F \subset I_1 \cup I_2 \cup \ldots. \cup I_m.$$

This completes the proof of the theorem.

Theorem 1.1.9. Every compact set of real numbers is bounded and closed.

Proof : Let E be a compact set of real numbers. We complete the proof by the following steps.

(I) For each x in E, let $I_x = (x-1, x+1)$ and let $\mathscr{F} = \{I_x : x \in E\}$. Then $\mathscr{F}$ is a family of open intervals which covers the set E. Since the set E is

compact, we can choose finite number of open intervals $I_{x_1}, I_{x_2}, \ldots, I_{x_n}$ from the family $\mathcal{F}$ such that

$$E \subset I_{x_1} \cup I_{x_2} \cup \ldots \cup I_{x_n} \quad \ldots \quad \ldots \quad (1.2)$$

Denote by a and b the least and the greatest of the numbers

$x_1-1, x_1+1, x_2-1, x_2+1, \ldots, x_n-1, x_n+1$.

Let $x \in E$. Then by (1.2) $x \in I_{xi}$ for some positive integer i $(1 \le i \le n)$ which gives that $a \le x_i - 1 < x < x_i + 1 \le b$. Hence E is bounded.

(II) Take any point x in E', where E' is the complement of E.

Let $y \in E$. Then $y \neq x$. Write $\delta_y = \frac{1}{3}|x-y|$ and $U_y = (y - \delta_y, y + \delta_y)$ $V_y = (x - + \delta_y, x + \delta_y)$. Then $x \in V_y$ and $y \in U_y$ and $V_y \cap U_y = \phi$.

Let $\mathcal{F} = \{U_y : y \in E\}$. Then $\mathcal{F}$ is a family of open intervals and it covers the set E. Since E is compact we can select finite number of open intervals

$U_{y_1}, U_{y_2}, \cdots, U_{y_n}$

from the family $\mathcal{F}$ such that

$$E \subset U_{y_1}, \cup U_{y_2}, \cup \cdots \cup, U_{y_n} \quad \ldots \quad \ldots \quad (1.3)$$

Let $V = \bigcap_{i=1}^{n} V_{y_i}$ and $W = \bigcup_{i=1}^{n} U_{y_i}$.

Then V is a neighd of x and $E \subset W$.

Let $z \in V$. Then $z \in V_{y_i}$ for $i = 1, 2, \ldots, n$.

Since $V_{y_i} \cap U_{y_i} = \phi$ $(i = 1, 2, \ldots, n)$, $z \notin U_{y_i}$ for $i = 1, 2, \ldots n$ and so $z \notin W$. From (1.3) it follows that $z \notin E$ and so $z \in E'$. Hence $V \subset E'$. So E' is a neighd of x. Since x is arbitrary it follows that E' is open and so E is closed.

Functions on subsets of real line.

Limit of a function. Let E be a nonempty subset of R and let $f: E \to R$. Let α be limit of point of the set E. We say that $f(x)$ tends to a finite limit l as x tends to α Over the set E if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$|f(x) - l| < \varepsilon$ for all $x \in E \cap \hat{N}(\alpha, \delta)$,

where $\hat{N}(\alpha, \delta) = \{x : x \in R \text{ and } 0 < |x - \alpha| < \delta\}$. We say that l is the limit of $f(x)$ as x tends to α over the set E and we write

$$\lim_{\substack{x \to \alpha \\ x \in E}} f(x) = l.$$

Continuity. Let $E \subset R$ and $f: E \to R$. The function f is said to be continuous at the point $\alpha \in E$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(\alpha)| < \varepsilon$ for all $x \in E \cap N(\alpha, \delta)$.

where $N(\alpha,\delta) = \{x : x \in R \text{ and } |x-\alpha| < \delta\}$.

If f is continuous at each point of E we say that f is continuous on E or simply continuous.

Note 1.3. Let $E \subset R$ and $f: E \to R$. Suppose that f is continuous at $\alpha \in E$. If α is *a* limit point of E, then clearly $f(x) \to f(\alpha)$ as $x \to \alpha$ over the set E.

Next, suppose that α is an isolated point of E. Then we can find *a* $\delta > 0$ such that $E \cap N(\alpha,\delta) = \{\alpha\}$. Choose any $\varepsilon > 0$. If $x \in E \cap N(\alpha,\delta)$, then $x = \alpha$ and so

$$|f(x) - f(\alpha)| = 0 < \varepsilon$$

This gives that f is continuous at α. Thus f is continuous at each isolated point of E

Theorem 1.20. Let E be *a* compact subset of R and let $f: E \to R$ be continuous. Then f is bounded on E and attains its bounds on E.

Proof : Let $\alpha \in E$. Since f is continuons at α, there is *a* $\delta_\alpha > 0$ such that

$$|f(x) - f(\alpha)| < 1 \text{ for all } x \in E \cap N(\alpha,\delta_\alpha) \quad \ldots \quad (1.4)$$

where $N(\alpha,\delta_\alpha) = \{x : x \in R \text{ and } |x - \alpha| < \delta_\alpha\}$.

Let $\Delta = \{N(\alpha,\delta_\alpha) : \alpha \in E\}$. Then Δ is *a* family of open imtervals which covers the set E. Since E *is* compact, we can select finite number of open intervals $N(\alpha_1, \delta_{\alpha_1}), N(\alpha_2, \delta_{\alpha_2}), \ldots, N(\alpha_n, \delta_{\alpha_n})$ from the family Δ such that

$$E \subset \cup_{i=1}^{n} N(\alpha_i, \delta_{\alpha_i}) \quad \ldots \quad \ldots \quad (1.5)$$

Let $k = max\{|f(\alpha_i)| + 1 : i = 1,2, \ldots, n\}$.

Take any $x \in E$. Then by (1.5), $x \in N(\alpha_i, \delta_{\alpha_i})$ for some i $(1 \le i \le n)$. So by (1.14)

$$|f(x) - f(\alpha_i)| < 1$$

or $|f(x)| \le |f(x) - f(\alpha_i)| + |f(\alpha_i)| < 1 + |f(\alpha_i)| \le k$.

Thus $|f(x)| \le k$ for all $x \in E$. So f is bounded on the set E.

Denote by M and m the lub and glb of $f(x)$ on E.

For each positive integer n, there is *a* point x_n in E such that

$$f(x_n) > M - \frac{1}{n}.$$

So for all n,

$$M - \frac{1}{n} < f(x_n) \le M \quad \ldots \quad \ldots \quad (1.6)$$

Since E is compact, it is bounded and closed. Again, $\{x_n\} \subset E$. So $\{x_n\}$ is bounded and hence it has *a* convergent subsequence $\{x_{n_k}\}$.

Let $\alpha = \lim_{k \to \infty} x_{n_k}$ Sinc E is closed $\alpha \in E$. From (1.6) we get

$$M - \frac{1}{n_k} < f\left(x_{n_k}\right) \leq M \text{ for all } k \qquad \ldots \qquad \ldots \qquad (1.7)$$

Since f is continuous at α, letting $k \to \infty$ in (1.7) we obtain $f(\alpha) = M$. Similarly we can show that $m = f(\beta)$ for some β in E.

Theorem 1.21. Let E be a compact subset of R and let $f : E \to R$ be continuous. Then the set $f(E)$ is compact.

Proof : Let Δ be a family of open intervals which covers the set $f(E)$. Take any point α in E. Then $f(\alpha) \in f(E)$. Since Δ is a cover of $f(E)$, there is an open interval I_α in Δ such that $f(\alpha) \in I_\alpha$. Clearly I_α is a neighd of $f(\alpha)$. So there is a positive number δ_α such that

$f(x) \in I_\alpha$ for all $x \in E \cap N(\alpha,\delta_\alpha)$ $\qquad \ldots \ldots$ (1.8)

where $N(\alpha,\delta_\alpha) = (\alpha-\delta_\alpha, \alpha+\delta_\alpha)$.

Let $\Delta^* = \{N(\alpha, \delta_\alpha) : \alpha \in E\}$. Then Δ^* is a family of open intervals which covers the set E. Since E is compact we can choose finite number of open intervals

$N(\alpha_1, \delta_{\alpha_1}), N(\alpha_2, \delta_{\alpha_2}), \ldots, N(\alpha_n, \delta_{\alpha_n})$ from the family Δ^* such that

$$E \subset \bigcup_{i=1}^{n} N\left(\alpha_i, \delta_{\alpha_i}\right) \qquad \ldots \qquad \ldots \qquad \ldots \qquad (1.9)$$

Now let $\beta \in f(E)$. Then there is a point α in E such that $\beta = f(\alpha)$. From (1.9) we see that $\alpha \in N\left(\alpha_i, \delta_{\alpha_i}\right)$ for some i $(1 \leq i \leq n)$. So by (1.8)

$$\beta = f(\alpha) \in I_{\alpha i} \subset \bigcup_{\nu=1}^{n} I_{\alpha_\nu}.$$

This gives that

$$f(E) \subset \bigcup_{\nu=1}^{n} I_{\alpha_\nu}.$$

Therefore the set $f(E)$ is compact.

Uniform Continuity. Let $E \subset R$ and $f : E \to R$. The function f is said to be uniformly countinuous on E if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$| f(x')-f(x'')| < \varepsilon$$

for all x', x'' in E with $| x'-x''| < \delta$.

Theorem 1.22. Let $E \subset R$ and $f : E \to R$ be uniformly continuous. Then f is continuous.

Proof. Let $\alpha \in E$. Choose any $\varepsilon > 0$. Since f is uniformly continuous on E, there is a $\delta > 0$ such that

$$| f(x') - f(x'')| < \varepsilon \ldots \qquad \ldots \qquad (1.10)$$

for all x', x'' in E with $|x'-x''| < \delta$.

Take any x in $E \cap N(\alpha,\delta)$, where $N(\alpha,\delta) = (\alpha - \delta, \alpha + \delta)$. Then

$$|x - \alpha| < \delta.$$

By (1.10) we have

$$|f(x) - f(\alpha)| < \varepsilon.$$

Therefore f is continuous at α. Since α is arbitrary, f is continuous.

Note 1.4. The converse of the above result is not true, that is, a function continuous on a set E may not be uniformly continuous. We show this by the following example.

Example 1.3.

Let $E = (0, 1)$. Define the function f on E as follows,

$$f(x) = \frac{1}{x} \text{ for all } x \in E.$$

Clearly f is continuous on E.

Assume that f is uniformly continuous on E. Choose any ε with $0 < \varepsilon < \frac{1}{2}$.

Then there is a $\delta > 0$ such that

$$|f(x') - f(x'')| < \varepsilon \qquad \ldots \qquad \ldots \qquad (1.11)$$

for all x', x'' in E with $|x' - x''| < \delta$.

Let n be a positive integer such that $n > \frac{1}{\sqrt{\delta}}$.

Take $x' = \frac{1}{n}$ and $x'' = \frac{1}{n+1}$.

Then $|x' - x''| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)} < \frac{1}{n^2} < \delta.$

So by (1.11) we get

$$|f(x') - f(x'')| < \varepsilon \qquad \ldots \qquad \ldots \qquad (1.12)$$

We have

$|f(x') - f(x'')| = |n - (n+1)| = 1$

This contradicts (1.12) because $\varepsilon < \frac{1}{2}$.

Hence f is not uniformly continuous on E.

Theorem 1.23. Let E be a compact subset of R and let $f : E \to R$ be continuous. Then f is uniformly continuous.

Proof : Choose any $\varepsilon > 0$. Take any $\alpha \in E$. Since f is continuous at α, there is a $\delta_\alpha > 0$ such that

$$| f(x) - f(\alpha)| < \frac{1}{2}\varepsilon \quad \ldots \quad \ldots \quad \ldots \quad (1.13)$$

for all $x \in E \cap N(\alpha, \delta_\alpha)$, where $N(\alpha,\delta_\alpha) = (\alpha-\delta_\alpha, \alpha + \delta_\alpha)$.
Let $x', x'' \in E \cap N(\alpha,\delta_\alpha)$. Then

$$| f(x') - f(x'') | \le | f(x') - f(\alpha)| + |f(x'') - f(\alpha)|$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon. \text{ [By (1.13)]}$$

Thus

$$| f(x') - f(x'')| < \varepsilon \quad \ldots \quad \ldots \quad (1.14)$$

for all x', x'' in $E \cap N(\alpha, \delta_\alpha)$.

Let $\Delta = \{N(\alpha,\delta_\alpha) : \alpha \in E\}$. Then Δ is *a* family of open intervals which covers the set E. Since E is compact there are finite number of open intervals $N(\alpha_1, \delta_{\alpha 1}), N(\alpha_2,\delta_{\alpha 2}), \ldots, N(\alpha_n,\delta_{\alpha n})$ in the family Δ such that

$$E \subset \bigcup_{i=1}^{n} N\left(\alpha_i, \delta_{\alpha_i}\right) \quad \ldots \quad \ldots \quad (1.15).$$

Let $\delta = \min \left\{\frac{1}{2}\delta_{\alpha_i} : i = 1, 2, 3, \cdots, n\right\}$. Then $\delta > 0$.

Take any x', x'' in E with $|x' - x''| < \delta$.
From (1.15) we see that $x' \in N(\alpha_i, \delta_{\alpha i})$ for some i $(1 \le i \le n)$. Then

$$| x'-\alpha_i| < \frac{1}{2}\delta_{a_i} \text{ and } | x''-\alpha_i | \le | x' - x''| + | x' - \alpha_i| < \delta + \frac{1}{2}\delta_{a_i} \le \delta_{k_i}$$

Thus x', x'' both belong to $E \cap N\left[\alpha_i, \delta_{\alpha_i}\right)$.

So by (1.14)

$$| f(x') - f(x'')| < \varepsilon.$$

Hence f is uniformly continuous on E.

Theorem 1.24. Let $f : [a, b] \to R$ be monotone. Then for each point x in (a, b), $f(x-0)$ & $f(x+0)$ exist and $f(a+0)$ & $f(b-0)$ exist. The set of the points of discontinuity of f in $[a, b]$ is countable.

Proof : Suppose that f is increasing on $[a,b]$. Take any α in (a,b) and let $E = \{y : y = f(x)$ for some x in $(a,\alpha)\}$ Clearly the set E is nonempty and bounded above. Denote by l the lub of the set E. Choose any $\varepsilon > 0$. There is a point x_1 in (a, α) such that $f(x_1) > l - \varepsilon$. Then clearly

$l - \varepsilon < f(x) \le l$ for all x in (x_1, α). This implies that $f(\alpha-0)$ exists and is equal to l. Similarly we can show that $f(\alpha+0)$ exists. Take u, v in (a, b) with $u < \alpha < v$. Then

$$f(u) \leq f(\alpha) \leq f(v).$$

Letting $u \to \alpha - 0$ and $v \to \alpha + 0$ we get

$$f(\alpha - 0) \leq f(\alpha) \leq f(\alpha+0).$$

Denote by D the set of all the points of discontinuity of f in (a,b). For α in D, *let* $I_\alpha = (f(\alpha-0). f(\alpha+0))$.

Take any α, β in D with $\alpha < \beta$. Then clearly $f(\alpha+0) \leq f(\beta-0)$.

This gives that $I_\alpha \cap I_\beta = \phi$. From each I_α choose *a* rational number λ_α and let $E = \{\lambda_\alpha : \alpha \in D\}$. Then E is countable. Define the mapping

$$g : D \to E \text{ as follows.}$$

$$g(\alpha) = \lambda_\alpha \text{ for } \alpha \in D.$$

Take any α, β in D with $\alpha \neq \beta$.

Without loss of generality we may take $\alpha < \beta$. Then $I_\alpha \cap I_\beta = \phi$. Since $\lambda_\alpha \in I_\alpha$ and $\lambda_\beta \in I_\beta$, $\lambda_\alpha \neq \lambda_\beta$, that is, $g(\alpha) \neq g(\beta)$. This gives that g is one-to-one Hence the set D is countable.

Theorem 1.25. Let f: $[a, b] \to R$ be decreasing. Then given any $\varepsilon > 0$ there is *a* subdivision $a = x_0 < x_1 < x_2 < ..... < x_n = b$ of $[a, b]$ such that

$$f(x_i+0) - f(x) < \varepsilon$$

for $x \in (x_i, x_{i+1})$ $(i = 0, 1, 2, ..., n-1)$.

Proof : Choose any $\varepsilon > 0$. We say that an interval $[\alpha, \beta] \subset [a, b]$ possesses the property (P) if there is *a subdivision*

$$\alpha = to < t_1 < t_2 < ... < t_n = \beta$$

of the interval $[\alpha, \beta]$ such that

$$f(t_i+) - f(x) < \varepsilon$$

for all x in (t_i, t_{i+1}) $(i = 0, 1, 2, ... , n - 1)$. Since $f(x) \to f(a +)$ there is *a* point α in $(a, b]$ such that

$$f(a+) - f(x) < \varepsilon \text{ for all } x \text{ in } (a, \alpha).$$

Let $a = t_0 < t_1 < t_2 < ... < t_n = \alpha$ be any subdivision of $[a, \alpha]$. Take any $x \in (t_i, t_{i+1})$. Then $x \in (a, \alpha)$, Since $f(t_i+) - f(x) \leq f(a+) - f(x)$ we have

$$f(t_i+) - f(x) < \varepsilon.$$

Thus the interval $[a, \alpha]$ possesses the Property (P).

Denote by E the set of all points α in $(a, b]$ such that $[a, \alpha]$ possesses the property (P). Clearly E is nonempty and bounded. So it has the least upper bound β (say). Then $a < \beta \leq b$. Let z be any point with $a < z < \beta$. There is an element α in E with $z < \alpha \leq \beta$. Since $[a, \alpha]$ possesses the property (P), the interval $[a, z]$ possesses the properly (P) and so $z \in E$.

Now we show that $\beta \in E$. Since $f(x) \to f(\beta-)$ as $x \to \beta-$, there is *a* point z_1 with $a < z_1 < \beta$ such that

$f(x) - f(\beta-) < \frac{1}{2}\varepsilon$ for all x in (z_1, β).

Let x, x' be two points with $z_1 < x' < x < \beta$. Since

$$f(x') - f(x) \leq f(x') - f(\beta-) < \frac{1}{2}\varepsilon,$$

letting $x' \to z_1 +$ we get

$$f(z_1+) - f(x) \leq \frac{1}{2}\varepsilon... \qquad ... \qquad (1.16).$$

Since $z_1 \varepsilon E$, there is a subdivision

$$a = t_0 < t_1 < ... < t_n = z_1$$

of $[a, z_1]$ such that

$$f(t_i+) - f(x) < \varepsilon$$

for all x in (t_i, t_{i+1}) $(i=0, 1, 2, 3, ... n-1)$.

Take $t_{n+1} = \beta$ and consider the subdivision

$$a = t_0 < t_1 < t_2 < ... < t_n < t_{n+1} = \beta.$$

Since $f(t_n+) - f(x) < \varepsilon$ for all x in (t_n, t_{n+1}) it follows that $[a, \beta]$ possesses the property (P). So $\beta \in E$.

Lastly we show that $\beta = b$ which will complete the proof.

Assume that $\beta < b$. There is a point z_2 with $\beta < z_2 < b$ such that $f(\beta+) - f(x) < \varepsilon$ for all x in (β, z_2).

Take $t_{n+2} = z_2$. Consider the subdivision $a = t_0 < t_1 < t_2 < ... < t_n < t_{n+1} < t_{n+2} = z_2$.

Since $f(t_{n+1}+) - f(x) < \varepsilon$ for all x in (t_{n+1}, t_{n+2}) we see that $[a, z_2]$ possesses the property (P). So $z_2 \in E$ which contradicts the fact that β is the lub of E. Hence $\beta = b$.

Semicontinuity. Let $E \subset R$ and let $f: E \to R$ f is said to be upper semicontinuous at $\alpha \in E$ if for every $\varepsilon > 0$ there is a $\delta_\alpha > 0$ such that

$$f(x) < f(\alpha) + \varepsilon \text{ for all } x \in E \cap N(\alpha,\delta),$$

where $N(\alpha,\delta) = (\alpha-\delta, \alpha+\delta)$.

f is said to be lower semicontinuous at $\alpha \in E$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$f(x) > f(\alpha) - \varepsilon \text{ for all } x\ (E \cap N(\alpha,\delta)).$$

From above it follows that if f is continuous at α, then it is both upper and lower semi continuous at α. Conversely if f is both upper and lower semicontinuous at α, then it is continuous at α.

Exercises—1.1

1. Let A and B be any two sets. Then
(i) $A/B = A \cap B'$ (ii) $A/(A/B) = A \cap B$.
2. Let A be any set and $\{E_n\}$ be any sequence of sets. Then

(i) $A \cap \left(\bigcup_{n=1}^{\infty} E_n\right) = \bigcup_{n=1}^{\infty}(A \cap E_n)$.

(ii) $A \cup \left(\bigcap_{n=1}^{\infty} E_n\right) = \bigcap_{n=1}^{\infty}(A \cup E_n)$.

(iii) $A/\left(\bigcup_{n=1}^{\infty} E_n\right) = \bigcap_{n=1}^{\infty}(A / E_n)$.

(iv) $A/\left\{\bigcup_{n=1}^{\infty}(A / E_n)\right\} = A \cap \left(\bigcap_{n=1}^{\infty} E_n\right)$.

(v) $A/\left(\bigcap_{n=1}^{\infty} E_n\right) = \bigcup_{n=1}^{\infty}(A / E_n)$.

(vi) $A/\left\{\bigcap_{n=1}^{\infty}(A / E_n)\right\} = \bigcup_{n=1}^{\infty}(A \cap E_n)$.

3. Let A and B be any two nonempty sets and $f : A \to B$ be *a* surjection. Let $\{A_n\} \subset A$ and $\{B_n\} \subset B$. Then
(i) $f(\cup_n A_n) = \cup_n f(A_n)$.
(ii) $f(\cap_n A_n) \subset \cap_n f(A_n)$.
(iii) $f^{-1}(\cup_n B_n) = \cup_n f^{-1}(B_n)$.
(iv) $f^{-1}(\cap_n B_n) = \cap_n f^{-1}(B_n)$.
4. Let E be *a* countable set and $\mathcal{F}$ be *a* family of all finite subsets of E. Then $\mathcal{F}$ is countable
5. Let E denote the set of all polynomials with rational coufficients. Then E is countable.
6. *A* number α (real or complex) is said to be an algebraic number if α is *a* root of some polynomial with integral coefficients. The set of all algebraic numbers is countable.
7. *A* function is said to be *a* polygonal function if its graph is *a* polygonal line. The set of all polygonal functions with finite number of rational corners is countable.
8. *A* set is said to be an isolated set if it has no limit point. Every isolated set on the real line is *a* countable set.
9. Let $E \subset R$ and $f : E \to R$. Then f is continuous if and only if one of the following holds.
(i) For any open set G in R, $f^{-1}(G) = E \cap W$, where W is open in R.

(ii) For any closed F set in R. $f^{-1}(F) = E \cap W$, where W is closed in R.

10. Let E be open in R and $f : E \to R$ be lower semicontinuous. Then for any real number σ, the set $\{x : x \in E \text{ and } f(x) > \sigma\}$ is open.

11. Let E be a closed set in R and $f : E \to R$ upper semicontinuous. Then for any real number σ the set

$\{x : x \in E \text{ and } f(x) \leq \sigma\}$ is closed.

Direeted sets and nets.

Directed sets. Let D be a nonempty set and "$\geq$" be a binary relation on the set D. The pair $(D, \geq)$ is said to be a directed set if the following conditions hold.

(i) $n \geq n$ for all n in D.

(ii) Let m, n, p belong to D. If $m \geq n$ and $n \geq p$, then $m \geq p$.

(iii) If m,n belong to D, then there is an element p in D with $p \geq m$ and $p \geq n$.

Let $(D, \geq)$ be a directed set and m, n belong to D. If $m \geq n$ and $m \neq n$ we write $m > n$. If $m \geq n$ we also write $n \leq m$.

Examples 1.4.

(1) Let N denote the set of all natural numbers and "$\geq$" denote the usual ordering on N. Then $(N, \geq)$ is a directed set.

(2) Let R denote the set of all real numbers and "$\geq$" denote the usual ordering on R. Then $(R, \geq)$ is a directed set.

(3) Let X denote a nonempty set and $\mathscr{F}$ denote the collection of all subsets of X. For A, B in $\mathscr{F}$, let $A \geq B$ if $A \supset B$. Then $(\mathscr{F}, \geq)$ is a directed set.

4. Let α be any real number. Denote by Δ the collection of all neigbovrhoods of α. For U, V in Δ, let $U \geq V$ if $U \subset V$. Then $(\Delta, \geq)$ is a directed set.

Nets. Let $(\Delta, \geq)$ be a directed set and X be a nonempty set. A mapping $s : \Delta \to X$ is called a net in X. For n in D, we write s_n in place of $s(n)$, the image of n under the mapping s. We denote by $\{s_n : n \in D, \geq\}$ or $\{s_n : n \in D\}$ or $\{s_n\}$ the net $s : D \to X$.

Examples 1.5.

(1) Let N denote the set of all natural numbers and "$\geq$" be its usual ordering. Then $(N, \geq)$ is a directed set. Let X denote the set of all rational numbers. Define $s : N \to X$ as follows.

$$s_n = \frac{n}{n+1} \text{ for all } n \in N.$$

Then $\{s_n : n \in N\}$ is *a* net in X. Clearly $\{s_n\}$ is *a* sequence. From this we see that every sequence is *a* net.

(2) Let R denote the set of all real numbers and let "$\geq$" denote the usual ordering on R. Then $(R, \geq)$ is *a* directed set. Take $X = R$ and define $s : R \to X$ as follows.

$$s_\alpha = \frac{\alpha}{1+\alpha^2} \text{ for all } \alpha \in R.$$

Then $\{s_\alpha : \alpha \in R\}$ is *a* net in X

(3) Let E be *a* nonempty set and let Δ denote the collection of all subsets of E. For A, B in Δ let $A \geq B$ if $A \supset B$. Then $(\Delta, \geq)$ is *a* directed set.

Define $s : \Delta \to \Delta$ as follows.

$$s(A) = A' \text{ for all } A \text{ in } \Delta,$$

where A' = complement of A *w.r.t* E

Then $\{s(A) : A \in \Delta\}$ is *a* net in Δ.

Nets of real numbers. Let $(D, \geq)$ be *a* directed set and X denote the set of all real numbers. *A* mapping $s : D \to X$ is called *a* net of real numbers.

A net $\{s_n : n \in D\}$ of real numbers is said to be bounded above if there is there is *a* real number k such that

$$s_n \leq k \text{ for all } n \text{ in } D.$$

A net $\{s_n : n \in D\}$ of real numbers is said to be bounded below if there is *a* real number k such that

$$s_n \geq k \text{ for all } n \in D.$$

A net $\{s_n : n \in D\}$ of real numbers is said to be bounded if it is bounded above *as* well *as* bounded below.

Convergent nets. Let $\{s_n : n \in D\}$ be *a* net of real numbers. $\{s_n\}$ is said to converge to the real number l if for every $\varepsilon > 0$ there is an element n_0 in D such that $|s_n - l| < \varepsilon$ for all n in D with $n \geq n_0$.

We say that l is the limit of the net $\{s_n\}$ and write

$$\lim_D s_n = l \text{ or } \lim_n s_n = l.$$

Monotone net. *A* net $\{s_n : n \in D\}$ of real numbers is said to be increasing [decreasing] if for any two elements m, n $(m \geq n)$ in D

$$s_m \geq s_n \ [s_m \leq s_n].$$

$\{s_n\}$ is said to be monotone if it is either increasing or decreasing.

Theorem 1.26. *A* monotone bounded net of real numbers is convergent.

Proof : Let $\{s_n : n \in D\}$ be *a* monotone bounded net of real numbers. Then there are real numbers a and b such that

$$a \leq s_n \leq b \text{ for all } n \text{ in } D. \quad \ldots \quad (1.17).$$

Suppose that $\{s_n\}$ is increasing.

Let $l = \sup \{s_n : n \in D\}$.

From (1.17) we have

$$a \leq l \leq b.$$

Choose any $\varepsilon > 0$. Then there is an element n_0 in D such that

$$s_{n_0} > l - \varepsilon. \quad \dots \quad \dots \quad (1.18)$$

Take any n in D with $n \geq n_0$. Since $\{s_n\}$ is increasing for any n in D with $n \geq n_0$

$$s_n \geq s_{n_0} \quad \dots \quad (1.19)$$

From (1.18) and (1.19) we get

$$l - \varepsilon < s_n \leq l$$

for all n in D with $n \geq n_0$.

This gives that

$$\lim_D s_n = l.$$

If $\{s_n : n \in D\}$ is decreasing, the proof is analogous.

Theorem 1.27. (Weierstrass' Approximation Theorem).

Let $f : [a, b] \to R$ be continuous and ε be any given positive number. Then there exists a polynomial p with real coefficients such that

$$|f(x) - p(x)| < \varepsilon \text{ for all } x \text{ in } [a, b].$$

Proof : (I) We first consider the case when $a = 0$ and $b = 1$.

We define the Bernstein polynomials $B_n(x)$ as follows.

$$B_n(x) = \sum_{k=0}^{n} {}^nC_k f\left(\frac{k}{n}\right)(1-x)^{n-k} x^k.$$

We have

$$\sum_{k=0}^{n} {}^nC_k (1-x)^{n-k} x^k = \{(1-x) + x\}^n = 1.$$

So

$$f(x) = \sum_{k=0}^{n} {}^nC_k f(x)(1-x)^{n-k} x^k$$

and

$$f(x) - B_n(x) = \sum_{k=0}^{n} {}^nC_k \left\{f(x) - f\left(\frac{k}{n}\right)\right\}(1-x)^{n-k} x^k.$$

Choose any $\varepsilon > 0$. Since f is continuous on $[0,1]$ it is uniformly continuous there. So we can find a positive number δ such that

$$|f(x') - f(x'')| < \frac{1}{2}\varepsilon$$

for all x', x'' in [0, 1] with $|x'-x''| < \delta$.
Take any x in [0, 1]. Let

$A = \{1, 2, 3. ..., n\}$, $P = \{k : k \in A$ and $\left|x - \frac{k}{n}\right| < \delta\}$ and $Q = A / P$;

and $M = \sup\{|f(x)| : a \le x \le b\}$.
We have

$$|f(x) - B_n(x)| \le \sum_{k\in P} {}^nC_k \left|f(x) - f\left(\frac{k}{n}\right)\right| (1-x)^{n-k} x^k$$

$$+ \sum_{k\in Q} {}^nC_k \left|f(x) - f\left(\frac{k}{n}\right)\right| (1-x)^{n-k} x^k$$

$$< \frac{1}{2}\varepsilon. \sum_{k\in P} {}^nC_k (1-x)^{n-k} x^k + 2M \sum_{k\in Q} {}^nC_k (1-x)^{n-k} x^k$$

$$< \frac{1}{2}\varepsilon. \sum_{k=0}^{n} {}^nC_k (1-x)^{n-k} x^k + \frac{2M}{n^2\delta^2} \sum_{k\in Q} {}^nC_k (nx-k)^2 (1-x)^{n-k} x^k$$

or

$$|f(x) - B_n(x)| < \frac{1}{2}\varepsilon + \frac{2M}{n^2\delta^2} \sum_{k=0}^{n} {}^nC_k (nx-k)^2 (nx-k)^{n-k} x^k \quad ... \quad (1.20)$$

For any two real numbers x, y we get

$$(y + x)^n = \sum_{k=0}^{n} {}^nC_k\, y^{n-x} x^k \quad ... \quad ... \quad (1.21)$$

Differentiating (1.21) *w.r.t.* x and muttiplying by x we obtain

$$n\, x\, (y+x)^{n-1} = \sum_{k=0}^{n} {}^nC_k\, k\, y^{n-k} x^k \quad ... \quad ... \quad (1.22).$$

Differentiating (1.22) *w.r.t.* x and multiplying by x we obtin

$$n\, x\, (y+x)^{n-1} + n(n-1)\, x^2\, (y+x)^{n-2} = \sum_{k=0}^{n} {}^nC_k\, k^2\, y^{n-k} x^k \quad ... \quad ... \quad (1.23)$$

Multiplying (1.21) by n^2x^2, (1.22) by $-2nx$ and (1.23) by 1 and then adding we have

$$\sum_{k=0}^{n} {}^nC_k \,(n^2x^2-2nxk + k^2)\, y^{n-k}x^k$$
$$= (y+x)^n.\ n^2x^2 - 2n^2x^2(y+x)^{n-1} + nx\,(y+x)^{n-1} + n(n-1)x^2(y+x)^{n-2}$$

Now taking $y = 1-x$ we obtain

$$\sum_{k=0}^{n} {}^nC_k \,(nx-k)^2\, (1-x)^{n-k}x^k$$
$$= n^2x^2-2n^2x^2 + nx + n^2x^2 - nx^2$$
$$= n\,(x-x^2) = \frac{n}{4}\left\{1-(1-2x)^2\right\} \le \frac{n}{4}. \qquad \dots \qquad \dots \qquad (1.24)$$

From (1.20) and (1.24) we get

$$|\, f(x) - B_n(x)\,| < \frac{1}{2}\varepsilon + \frac{2M}{n^2\delta^2}\cdot\frac{n}{4} = \frac{1}{2}\varepsilon + \frac{M}{2\delta^2}\cdot\frac{1}{n}.$$

Now choose positive integer n_0 such that $n_0 \ge \dfrac{M}{\delta^2\varepsilon}$.

Then for any $n \ge n_0$

$|\, f(x) - B_n(x)\,| < \varepsilon$ for all x in $[0, 1]$.

(II) Now consider the case when $[a, b] \ne [0, 1]$.

Let $y = \dfrac{x-a}{b-a}$. Then $x = a + (b-a)\,y$.

As x varies in $[a, b]$, y varies in $[0,1]$ and conversely when y varies in $[0, 1]$, x varies in $[a, b]$.

We have

$f(x) = f(a+(b-a)\,y) = g(y)$ (say).

Clearly g is continuous on $[0,1]$.

Choose any $\varepsilon > 0$.

By case *I*, there is a polynomial $Q(y)$ such that

$|\, g(y) - Q(y)\,| < \varepsilon$ for all y in $[0,1]$

or $\left| f(x) - Q\left(\dfrac{x-a}{b-a}\right)\right| < \varepsilon$ for all x in $[a, b]$.

Write $p(x) = Q\left(\dfrac{x-a}{b-a}\right)$. Then $p(x)$ is a polynomial in x and

$|\, f(x) - p(x)\,| < \varepsilon$ for all x in $[a,b]$.

CHAPTER—II

SETS RINGS AND σ-RINGS

Definition 2.1. *A* nonempty class $\mathcal{R}$ of sets is said to be *a* ring if for every pair of sets E *and* F in $\mathcal{R}$, $E \cup F$ and E / F are in $\mathcal{R}$.

Clearly every ring contains the void set. Let E and F be any two members of the ring $\mathcal{R}$. Then E/F belongs to $\mathcal{R}$ and so $E/(E/F)$ belongs to $\mathcal{R}$. Since $E \cap F = E|(E \mid F)$ it follows that $E \cap F$ belongs to $\mathcal{R}$.

If $E_1, E_2, ..., E_n$ are members of the ring $\mathcal{R}$, then by the principle of induction we see that $\bigcup_{i=1}^{n} E_i$ and $\bigcap_{i=1}^{n} E_i$ belong to $\mathcal{R}$.

Examples 2.1. Let X be a nonempty set.

(1) If $\mathcal{R}$ denote the class of all finite subsets of X, then $\mathcal{R}$ is *a* ring.

(2) If $\mathcal{R}$ denote the class of all countable subsets of X, then $\mathcal{R}$ is *a* ring.

(3) Let $\mathcal{R}$ denote the class of all subsets of X. Then $\mathcal{R}$ is *a* ring.

Definition 2.2. *A* nonempty class $\mathcal{S}$ of sets is said to be *a* σ-ring if $\mathcal{S}$ is *a* ring and for any sequence $\{E_i\}_{i=1}^{\infty}$ of sets in $\mathcal{S}$, $\bigcup_{i=1}^{\infty} E_i$ belongs to $\mathcal{S}$.

Let $\mathcal{S}$ be *a* σ-ring and $\{E_i\}_{i=1}^{\infty}$ be *a* sequence of sets in $\mathcal{S}$. Write $E = \bigcup_{i=1}^{\infty} E_i$. Then $E \in \mathcal{S}$. Also $E/E_i \in \mathcal{S}$ for $i = 1,2,3, ...$ so $\bigcup_{i=1}^{\infty} (E \setminus E_i) \in \mathcal{S}$.

Since $\bigcap_{i=1}^{\infty} E_i = E/\left\{\bigcup_{i=1}^{\infty}(E / E_i)\right\}$ we see that $\bigcap_{i=1}^{\infty} E_i \in \mathcal{S}$.

In Examples 2.1 $\mathcal{R}$ is *a* σ-ring in the cases (2) and (3).

Theorem 2.1. The intersection of *a* nonempty family of rings [σ-rings] is *a* ring [σ-ring].

Proof : Let $\mathcal{F} = \{\mathcal{R}_\alpha : \alpha \in \Delta\}$ be *a* nonempty family of rings and let $\mathcal{R} = \cap \{\mathcal{R}_\alpha : \alpha \in \Delta\}$. Let E, F be any two members of $\mathcal{R}$. Then E, F belong to $\mathcal{R}_\alpha$ for every α in Δ. Since each $\mathcal{R}_\alpha$ is *a* ring $E \cup F$ and $E \setminus F$ belong to $\mathcal{R}_\alpha$ for every α in Δ. So $E \cup F$ and $E \setminus F$ belong to $\cap\{\mathcal{R}_\alpha : \alpha \in \Delta\} = \mathcal{R}$. Hence $\mathcal{R}$ is *a* ring.

Next, let $\mathcal{F} = \{\mathcal{S}_\alpha : \alpha \in \Delta\}$ be *a* non empty family of σ-rings and let $\mathcal{S} = \cap\{\mathcal{S}_\alpha : \alpha \in \Delta\}$. Since each $\mathcal{S}_\alpha$ is *a* ring by above case $\mathcal{S}$ is *a* ring.

Now let $\{E_i\}_{i=1}^{\infty}$ be any sequence of sets in $\mathcal{S}$. Then $\{E_i\}_{i=1}^{\infty} \subset \mathcal{S}_\alpha$ for every

α in Δ. Since each $\mathcal{S}_\alpha$ is *a* σ-ring, $\bigcup_{i=1}^{\infty} E_i \in \mathcal{S}_\alpha$ for every α in Δ and so

$\bigcup_{i=1}^{\infty} E_i \in \cap\{\mathcal{S}_\alpha : \alpha \in \Delta\} = \mathcal{S}$. Hence $\mathcal{S}$ is *a* σ-ring.

Definition 2.3. Let $\mathcal{E}$ be *a* nonempty class of sets.

(I) Denote by $\mathcal{F}$ the family of all rings containing the class $\mathcal{E}$. Then clearly $\mathcal{F}$ is nonempty because the class of all sets is *a* ring containg the class $\mathcal{E}$.

Let $\mathcal{R}_0 = \cap\{\mathcal{R} : \mathcal{R} \in \mathcal{E}\}$.

By Theorem 2.1, $\mathcal{R}_0$ is *a* ring. Since each $\mathcal{R}$ in $\mathcal{F}$ contains the class $\mathcal{E}$, it follows that $\mathcal{E} \subset \mathcal{R}_0$.

We denote $\mathcal{R}_0$ by $\mathcal{R}(\mathcal{E})$ and call $\mathcal{R}(\mathcal{E})$ the ring generated by the class $\mathcal{E}$.

(II) Denote by $\mathcal{F}$ the family of all σ-rings containg the class $\mathcal{E}$. Then $\mathcal{F}$ is nonempty.

Let $\mathcal{S}_0 = \cap\{\mathcal{S} : \mathcal{S} \in \mathcal{F}\}$.

By Theorem 2.1, $\mathcal{S}_0$ is *a* σ ring. Since each $\mathcal{S}$ in $\mathcal{F}$ contains the class $\mathcal{E}$, it follows that $\mathcal{E} \subset \mathcal{S}_0$.

We denote $\mathcal{S}_0$ by $\mathcal{S}(\mathcal{E})$ and call $\mathcal{S}(\mathcal{E})$ the σ-ring generated by the class $\mathcal{E}$.

Theorem 2.2. Let $\mathcal{E}$ be *a* nonempty class of sets.

(I) Every set in $\mathcal{R}(\mathcal{E})$ can be covered by finite number of sets in $\mathcal{E}$.

(II) Every set in $\mathcal{S}(\mathcal{E})$ can be covered by *a* countable class of sets in $\mathcal{E}$.

Proof. (I) Let $\mathcal{R}_0$ denote the class all sets which can be covered by finite number of sets in $\mathcal{E}$. Take any two sets E and F from $\mathcal{R}_0$. Then

$E \subset \bigcup_{i=1}^{m} E_i$ and $F \subset \bigcup_{i=1}^{n} F_i$, where E_i, F_i belong to the class $\mathcal{E}$.

We have

$E \cup F \subset E_1 \cup E_2 \cup \ldots \cup E_n \cup F_1 \cup \ldots \cup F_2 \cup \ldots \cup F_n$

and $E \setminus F \subset E \subset E_1 \cup E_2 \cup \ldots \cup E_m$.

This gives that $E \cup F$ and $E \setminus F$ belong to $\mathcal{R}_0$ Hence $\mathcal{R}_0$ is *a* ring. Clearly R_0 contains the class $\mathcal{E}$. Since $\mathcal{R}(\mathcal{E})$ is the smallest ring containing the class $\mathcal{E}$ it follows that $\mathcal{R}(\mathcal{E}) \subset \mathcal{R}_0$. Hence every set in $R(\mathcal{E})$ can be covered by finite number of setsfrom the class $\mathcal{E}$.

(II) Denote by $\mathcal{S}_0$ the class of all sets which can be covered by countable class of sets from $\mathcal{E}$.

Take any two sets E and F in $\mathcal{S}_0$. Then

$E \subset \bigcup_{i=1}^{\infty} E_i$ and $F \subset \bigcup_{i=1}^{\infty} F_i$,

where E_i, F_i belong to $\mathcal{E}$.

Since $E \cup F \subset \bigcup_{i=1}^{\infty} (E_i \cup F_i)$ and $E \setminus F \subset E \subset \bigcup_{i=1}^{\infty} F_i$, it follows that $\mathcal{S}_0$ is *a* ring.

Now let $\{E_i\}_{i=1}^{\infty}$ be any seqnence of sets in $\mathcal{S}_0$. Then for each i,

$$E_i \subset \bigcup_{j=1}^{\infty} E_{ij}, \text{ where } E_{ij} \in \mathcal{E}.$$

Let $E = \bigcup_{i=1}^{\infty} E_i$. We have

$$E \subset \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} E_{ij},$$

Since $\{E_{ij} : i = 1, 2, 3, \ldots \text{ and } j = 1, 2, 3, \ldots\}$ is *a* countable class it follows that $E \in \mathcal{S}_0$. So $\mathcal{S}_0$ is *a* σ-ring. Clearly $\mathcal{E} \subset \mathcal{S}_0$. So $\mathcal{S}_0$ is *a* σ-ring containg the class $\mathcal{E}$. Since $\mathcal{S}(\mathcal{E})$ is the smallest σ-ring containg the class $\mathcal{E}$ we have $\mathcal{S}(\mathcal{E}) \subset \mathcal{S}_0$.

Hence every set in $\mathcal{S}(\mathcal{E})$ can be covered by countable class of sets from the class $\mathcal{E}$.

Definition 2.4. *A* nonempty class $\mathcal{P}$ of sets is said to be *a* semiring if the following conditions hold.

(i) The void set ϕ belongs to $\mathcal{P}$.

(ii) If E, F belong to $\mathcal{P}$, then $E \cap F$ belongs to $\mathcal{P}$.

(iii) If E, F belong to $\mathcal{P}$, then $E \setminus F$ can be expressed as

$$E \setminus F = \bigcup_{i=1}^{n} E_i$$

where $E_i \in \mathcal{P}$ and $E_i \cap E_j = \phi$ for $i \neq j$.

Example 2.2. Let X denote *a* nonempty set. If $\mathcal{P}$ denotes the class of all finite sets, then $\mathcal{P}$ is *a* semiring.

Example 2.3. Let $\mathcal{P}$ denote the class of all sets of the form $[a, b)$, where a and b are real numbers and $a \leq b$. Then $\mathcal{P}$ is *a* semiring.

Soln. If $a = b$, then $[a, b) = \phi$ which gives that the void set ϕ belongs to $\mathcal{P}$. Let E and F be any two members of $\mathcal{P}$. Then $E = [a, b)$ and $F = [c, d)$, where a, b, c, d are real numbers and $a \leq b$, $c \leq d$. If $E \cap F = \phi$, then $E \cap F \in \mathcal{P}$. Suppose that $E \cap F \neq \phi$. Let $\alpha = \max\{a, c\}$ and $\beta = \min\{b, d\}$. Then $E \cap F = [\alpha, \beta)$. So $E \cap F \in \mathcal{P}$.

Now write $D = E \cap F$. Suppose that $E \setminus F \neq \phi$. We have $E \setminus F = E \setminus D$. Following cases come up for consideration.

(i) $a < \alpha$, $\beta < b$ (ii) $a < \alpha$, $\beta = b$.

(iii) $a = \alpha$, $\beta < b$.

Case (i) write $E_1 = [a, \alpha)$ and $E_2 = [\beta, b)$. Then $E_1, E_2 \in \mathcal{P}$. We have $E \setminus F = E \setminus D = E_1 \cup E_2$ and $E_1 \cap E_2 = \phi$.

Case (ii) Write $E_1 = [a, \alpha)$. Then $E_1 \in \mathcal{P}$.

We have $E \setminus F = E \setminus D = E_1$.

Case (iii) Write $E_2 = [\beta, b)$. Then $E_2 \in \mathcal{P}$.

We have $E \setminus F = E \setminus D = E_2$.

From above considerations it follows that $\mathcal{P}$ is *a* semiring.

Note 2.1. The semiring in example 2.3 is not *a* ring.

Let $E = [0,1)$ and $F = [2, 3)$.

Then $E, F \in \mathcal{P}$. Clearly $E \cup F$ does not belong to $\mathcal{P}$. So $\mathcal{P}$ is not *a* ring.

From above we see that *a* semiring may not be *a* ring. It is easy to see that every ring is *a* semiring.

Theorem 2.3. Let $\mathcal{P}$ be *a* semiring and let $\mathcal{R}_0$ denote the class of all unions of fimite number of mutually disjoint sets in $\mathcal{P}$. Then $\mathcal{R}_0$ is *a* ring and $\mathcal{R}_0 = \mathcal{R}(\mathcal{P})$. 247103

Proof : Let E and F be any two sets in $\mathcal{R}_0$. Then we can express E and F as follows.

$$\left.\begin{array}{l} E = \bigcup_{i=1}^{m} E_i \text{ and } F = \bigcup_{i=1}^{n} F_i, \\ \text{where } E_i, F_i \text{ belong to } \mathcal{P} \text{ and} \\ E_i \cap E_j = \phi,\ F_i \cap F_j = \phi \text{ for } i \neq j. \end{array}\right\} \quad \ldots \quad (2.1)$$

(I) Suppose that $E \cap F = \phi$. Write

$$E_{m+i} = F_i = (i = 1.2\ldots., n).$$

Then $E \cup F = \bigcup_{i=1}^{m+n} E_i$, where the sets $E_1, E_2, E_3, \ldots, E_{m+n}$ belong to $\mathcal{P}$ and they are mutually disjoint. So $E \cup F \in \mathcal{R}_0$.

(II) Suppose that $E \cap E \neq \phi$. We have

$$E \cap F = \bigcup_{i=1}^{m} (E_i \cap F) = \bigcup_{i=1}^{m} A_i,$$

where $A_i = E_i \cap F = \bigcup_{j=1}^{n} (E_i \cap F_j) = \bigcup_{j=1}^{n} B_j$ (say).

Since E_i, F_j belong to $\mathcal{P}$, $B_j = E_i \cap F_j \in \mathcal{P}$. Clearly the sets $B_1, B_2, \ldots, B_n$ are nutually disjoint. So $A_i \in \mathcal{R}_0$ $(i = 1, 2, \ldots, m)$. Since $A_i \cap A_k = \phi$ for $i \neq k$, by case *I*, $E \cap F \in \mathcal{R}_0$.

(III) We have $E/F = \bigcup_{i=1}^{m} (E_i / F) = \bigcup_{i=1}^{m} A_i$,

where $A_j = E_i \setminus F = E_i / \left(\bigcup_{j=1}^{n} F_j\right) = \bigcap_{j=1}^{n}\left(E_i / F_j\right)$

$$= \bigcap_{j=1}^{n} B_j \text{ (say)}.$$

Now $B_j = E_i / F_j$. Since E_i, F_j belong to $\mathscr{P}$, we can write $E_i / F_j = D_1 \cup D_2 \cup \ldots \cup D_p$, where $D_1, D_2, \ldots, D_p$ belong to $\mathscr{P}$ and they are mutually disjoint. So $B_j \in \mathscr{R}_0$. Clearly $B_r \cap B_k = \phi$ for $r \neq k$. Hence by case II, $A_i \in \mathscr{R}_0$. Also the sets A_1, A_2, A_m are pairwise disjoint. So by case I, $E / F \in \mathscr{R}_0$.

(IV) We have $E \cup F = (E \setminus F) \cup (E \cap F) \cup (F \setminus E)$.

By cases *I, II, III*, it follows that $E \cup F \in \mathscr{R}_0$.

From above discussion we see that for any two sets E and F in $\mathscr{R}_0$, $E \cup F$ and E/F belong to $\mathscr{R}_0$. Hence $\mathscr{R}_0$ is *a* ring.

Clearly $\mathscr{P} \subset R_0$ which gives that $R(\mathscr{P}) \subset R_0$. Again, $R(\mathscr{P})$ is *a* ring containg $\mathscr{P}$; so it contains all finite unions of mutually disjoint sets in $\mathscr{P}$ which gives that $R_0 \subset R(\mathscr{P})$. Therefore $R_0 = R(\mathscr{P})$.

Definition 2.5. Let $\mathscr{E}$ be *a* nonempty class of sets and A be *a* fixed set. We denote by $\mathscr{E} \cap A$ the class of sets of the form $E \cap A$, where $E \in \mathscr{E}$.

If $\mathscr{R}$ is *a* ring, then it is easy to see that $\mathscr{R} \cap A$ is *a* ring. Again, if $\mathscr{S}$ is *a* σ-ring, then $\mathscr{S} \cap A$ is *a* σ-ring.

Theorem 2.4. Let $\mathscr{E}$ be *a* nonempty class of sets and A be *a* fixed set. Then

$$\mathscr{S}(\mathscr{E}) \cap A = \mathscr{S}(\mathscr{E} \cap A).$$

Proof : Clearly $\mathscr{E} \cap A \subset \mathscr{S}(\mathscr{E}) \cap A$ and $\mathscr{S}(\mathscr{E}) \cap A$ is *a* σ-ring. So we get

$$\mathscr{S}(\mathscr{E} \cap A) \subset \mathscr{S}(\mathscr{E}) \cap A \quad \ldots \quad \ldots \quad (2.2).$$

To show the reverse inclusion relation we proceed as follows.

Denote $\mathscr{B}$ the class of all sets of the form $B = D \cup (F \setminus A)$, where $D \in \mathscr{S}(\mathscr{E} \cap A)$ and $F \in \mathscr{S}(\mathscr{E})$. We show that $\mathscr{B}$ is *a* σ-ring.

Let B_1 and B_2 be any two sets of $\mathscr{R}$. Then $B_i = D_i \cup (F_i \setminus A)$, where $D_i \in \mathscr{S}(\mathscr{E} \cap A)$ and $F_i \in \mathscr{S}(\mathscr{E})$ $(i = 1, 2)$. We have

$$B_1 \cup B_2 = (D_1 \cup D_2) \cup \{(F_1 \mid A) \cup (F_2 \mid A)\}$$
$$= (D_1 \cup D_2) \cup \{(F_1 \cup F_2) \setminus A\}.$$

and $B_1 \setminus B_2 = \{D_1 \cup (F_1 \mid A)\} \mid \{D_2 \cup (F_2 \mid A)\}$

$$= (D_1 \mid D_2) \cup \{(F_1 \mid F_2) \mid A\} \ (\because D_i \subset A)$$

So $B_1 \cup B_2$ and $B_1 \setminus B_2$ belong to $\mathscr{B}$. Therfore $\mathscr{B}$ is a ring.

Next, let $\{B_i\}_{i=1}^{\infty}$ be any sequence of sets in $\mathcal{B}$. Then
$B_i = D_i \cup (F_i \mid A)$ $(i = 1,2,3, ...)$,
where $D_i \in \mathcal{S}(\mathcal{E} \cap A)$ and $F_i \in \mathcal{S}(\mathcal{E})$ $(i = 1, 2, 3,...)$.
We have

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} \{D_i \cup (Fi \setminus A)\}$$

$$= \left(\bigcup_{i=1}^{\infty} D_i\right) \cup \left\{\left(\bigcup_{i=1}^{\infty} F_i\right) \| A\right\}.$$

This shows that $\bigcup_{i=1}^{\infty} B_i$ belongs to $\mathcal{B}$.

Hence $\mathcal{B}$ is *a* σ-ring.

Let $E \in \mathcal{S}$. We can write

$$E = (E \cap A) \cup (E \setminus A).$$

This gives that $E \in \mathcal{B}$. So $\mathcal{E} \subset \mathcal{B}$ which again gives that $\mathcal{S}(\mathcal{B}) \subset \mathcal{B}$. Hence

$$\mathcal{S}(\mathcal{E}) \cap A \subset \mathcal{B} \cap A. \qquad \ldots \qquad \ldots \qquad (2.2)$$

Let $B \in \mathcal{B}$. Then $B = D \cup (F \setminus A)$, where $D \in \mathcal{S}(\mathcal{E} \cap A)$ and $F \in \mathcal{S}(\mathcal{E})$. So $B \cap A = D \cap A = D$ because $D \subset A$. This gives that $B \cap A \in \mathcal{S}(\mathcal{E} \cap A)$ and So.

$$\mathcal{B} \cap A \subset \mathcal{S}(\mathcal{E} \cap A) \qquad \ldots \qquad \ldots \qquad (2.3)$$

From (2.2) and (2.3) we get

$$\mathcal{S}(\mathcal{E}) \cap A \subset \mathcal{S}(\mathcal{E} \cap A). \qquad \ldots \qquad \ldots \qquad (2.4)$$

From (2.1) and (2.4) we obtain

$$\mathcal{S}(\mathcal{E}) \cap A = \mathcal{S}(\mathcal{E} \cap A).$$

Definition 2.6. Let $\{E_n\}$ be *a* sequence of sets. We define the set E^* as follows.

E^* consists of all those elements which belong to infinity of sets E_n. We call E^* the upper limit or limit superior of the sequence $\{E_n\}$ and write

$$E^* = \lim_{n \to \infty} \sup E_n \text{ or } E^* = \overline{\lim_{n \to \infty}} E_n.$$

Next, we define another set E_* as follows.

E_* consists of all those elements which belong to all sets E_n except *a* finite number. We call E_* the lowerlimit or limit inferior of the sequence $\{E_n\}$ and we write

$$E_* = \lim_{n \to \infty} \inf E_n \text{ or } E_* = \underline{\lim_{n \to \infty}} E_n.$$

From above definitions we see that

$E^* = \{x : x \in E_n$ for infinity of $n\}$ and

$E_* = \{x : x \in E$ for all n except a finite number$\}$.

It is clear that $E_* \subset E^*$.

If $E^* = E_* = E$ (say), then we say that the sequence $\{E_n\}$ converges to the set E and we werite

$$E = \lim_{n\to\infty} E_n.$$

Monotone sequence of sets. *A* sequence $\{E_n\}$ of sets is said to be increasing or expanding if $E_n \subset E_{n+1}$ for every n.

A sequence $\{E_n\}$ of sets is said to be decreasing or contracting if $E_n \supset E_{n+1}$ for every n.

A sequence $\{E_n\}$ of sets is said to be monotone if it is either increasing or decreasing.

Theorem 2.5. Let $\{E_n\}$ be any sequence of sets. Then

$$E^* = \lim_{n\to\infty} \sup E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$$

and $$E_* = \lim_{n\to\infty} \inf E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n.$$

Proof : (I) Let $x \in E^*$. Then x belongs to E_n for infinity of n. This implies

that $$x \in \bigcup_{n=k}^{\infty} E_n$$

for every positive intger k. Therefore

$$x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n. \quad \ldots \quad \ldots \quad (2{\cdot}5)$$

Again let x be any element of the right hand set of (2.5). Then

$$x \in \bigcup_{n=k}^{\infty} E_n.$$

for every positive integer k. This implies that x belongs to E_n for infinity of n.

So $x \in E^*$. Hence we obtain

$$E^* = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n.$$

(II) Next, let $x \in E_*$. Then x belongs to E_n for all n except a finite number. This gives that there is a positive integer k such that $x \in E_n$ for all $n \geq k$. Thus

$x \in \bigcap_{n=k}^{\infty} E_n$ and hence

$$x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n \quad \ldots \quad \ldots \quad (2{\cdot}6)$$

Now let x belong to the right hand set of (2.6). Then

$$x \in \bigcap_{n=k}^{\infty} E_n$$

for some positive integer k which gives that $x \in E_n$ for all $n \geq k$. So $x \in E_*$. Therefore we obtain

$$E_* = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$$

Theorem 2.6. Every monotone sequence $\{E_n\}$ of sets is convergent.

Proof : (I) Suppose that $\{E_n\}$ is an increasing sequence of sets. Then $E_1 \subset E_2 \subset E_3 \subset$ ....

Let $E = \bigcup_{n=1}^{\infty} E_n$. For every positive integer k we have

$$\bigcup_{n=k}^{\infty} E_n = \bigcup_{n=1}^{\infty} E_n = E.$$

So $E^* = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n = E.$

Again, for any positive integer k.

$$\bigcap_{n=k}^{\infty} E_n = E_k \ (\because\ E_k \subset E_n \text{ for } n \geq k).$$

So $E_* = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n = \bigcup_{k=1}^{\infty} E_k = E.$

Thus $E^* = E = E_*$. Hence $\{E_n\}$ is convergent and

$$\lim_{n \to \infty} E_n = \bigcup_{k=1}^{\infty} E_k.$$

(II) Suppose that $\{E_n\}$ is decreasing.

Then $E_1 \supset E_2 \supset E_3 \supset$ ....

For any positive integer k, we have

$$\bigcup_{n=k}^{\infty} E_n = E_k$$

So $E^* = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_k = \bigcap_{k=1}^{\infty} E_k = A(say).$

Again, for any positive integer k,

$$\bigcap_{n=k}^{\infty} E_n = \bigcap_{n=1}^{\infty} E_n = A.$$

So $E_* = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n = A.$

Thus $E^* = A = E^*$.

Hence the sequence $\{E_n\}$ is convergent and

$$\lim_{n \to \infty} E_n = \bigcap_{k=1}^{\infty} E_k.$$

Problem 2.1. Let $\{E_n\}$ be *a* monotone sequence of sets and A be any set. Then prove the following.

(1) $A \setminus \left(\lim\limits_{n\to\infty} E_n \right) = \lim\limits_{n\to\infty} (A \setminus E_n)$.

(2) $\left(\lim\limits_{n\to\infty} E_n \right) \setminus A = \lim\limits_{n\to\infty} (E_n \setminus A)$.

(3) $A \cap \left(\lim\limits_{n\to\infty} E_n \right) = \lim\limits_{n\to\infty} (A \cap E_n)$

(4) $A \cup (\lim E_n) = \lim\limits_{n\to\infty} (A \cup E_n)$

Definition 2.7. A non-empty class $\mathcal{M}$ of sets is said to be *a* monotone class if for every monotone sequence $\{E_n\}$ of sets in $\mathcal{M}$ $\lim E_n \in \mathcal{M}$.

Every σ-ring is cleary *a* monotone class. Let $\mathcal{E}$ be any nonempty class of sets. The monotone class generated by $\mathcal{E}$ is the intersection of all monotone classes containing the class $\mathcal{E}$ and is denoted by $\mathcal{M}(\mathcal{E})$. A ring which is also a monotone class is called *a* monotone ring.

Theorem 2.7. A monotone ring is *a* σ-ring.

Froof : Let $\mathcal{M}$ be *a* monotone ring. Take any sequence $\{E_n\}$ of sets in $\mathcal{M}$. We define the sets $A_1, A_2, A_3, \ldots$ as follows.

$A_1 = E_1$, $A_2 = E_1 \cup E_2$, $A_3 = E_1 \cup E_2 \cup E_3$, ...Then $\{A_n\}$ is *a* monotone increasing sequence and $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} E_n$.

Since $\mathcal{M}$ is *a* ring, $A_n \in \mathcal{M}$ for every n. Again, since $\mathcal{M}$ is *a* monotone class, $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$, that is, $\bigcup_{n=1}^{\infty} E_n \in \mathcal{M}$. Hence $\mathcal{M}$ is *a* σ-ring.

Theorem 2.8. If $\mathcal{R}$ is *a* ring, then $\mathcal{M}(\mathcal{R}) = \mathcal{S}(\mathcal{R})$.

Proof : Let $\mathcal{R}$ be *a* ring. Since *a* σ-ring is a monotone class and $\mathcal{R} \subset \mathcal{S}(\mathcal{R})$ we have

$$\mathcal{M}(\mathcal{R}) \subset \mathcal{S}(\mathcal{R}) \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2.7)$$

To show the reverse inclusion relation we proceed as follows.

For any set F, let $\mathcal{K}(F)$ denote class of all sets E such that $E \setminus F$, $F \setminus E$ and $E \cup F$ belong to $\mathcal{M}(\mathcal{R})$. It is clear that

$$E \in \mathcal{K}(F) \Leftrightarrow F \in \mathcal{K}(E).$$

(I) Suppose that $\mathcal{K}(F)$ is not empty.

Let $\{E_n\}$ be any monotone sequence in $\mathcal{K}(F)$. Then $\{E_n \setminus F\}$, $\{F \setminus E_n\}$ and $\{F \cup E_n\}$ are monotone sequences in $\mathcal{M}(\mathcal{R})$. We have

$$\left(\lim_{n\to\infty} E_n\right)\setminus F = \lim_{n\to\infty} (F_n \setminus F) \in \mathcal{M}\ (\mathcal{R}),$$

$$F\setminus\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} (F \setminus F_n) \in \mathcal{M}\ (\mathcal{R}),$$

$$F\cup\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} (F \cup F_n) \in \mathcal{M}\ (\mathcal{R}).$$

This shows that $\lim E_n \in \mathcal{K}\,(F)$ and so $\mathcal{K}\,(F)$ is *a* monotone class.

(II) Let F belong to $\mathcal{R}$ and keep it fixed. If $E \in \mathcal{R}$, then $E \mid F$, $F \setminus E$ and $E \cup F$ also belong to $\mathcal{R}$. Since $\mathcal{R} \subset \mathcal{M}\,(\mathcal{R})$, it follows that $E \in \mathcal{K}\,(F)$. By case *I*, $\mathcal{K}\,(F)$ is *a* monotone class. Also $\mathcal{R} \subset \mathcal{K}\,(F)$.

Therefore,

$$\mathcal{M}\,(\mathcal{R}) \subset \mathcal{K}\,(F).$$

(III) Let E and F be any two sets in $\mathcal{M}\,(\mathcal{R})$. By case *II*, $E \in \mathcal{K}\,(F)$. This gives that $E \cup F$, $E \setminus F$ and $F \setminus E$ belong to $\mathcal{M}\,(\mathcal{R})$. Hence $\mathcal{M}\,(\mathcal{R})$ is *a* ring and by Theorem 2·7, $\mathcal{M}\,(\mathcal{R})$ is *a* σ-ring. So

$$\mathcal{S}\,(\mathcal{R}) \subset \mathcal{M}\,(\mathcal{R}) \qquad \ldots \qquad \ldots \qquad (2{\cdot}8)$$

From (2.7) and (2.8) we obtain

$$\mathcal{M}\,(\mathcal{R}) = \mathcal{S}\,(\mathcal{R}).$$

Corollary 2.8.1. It $\mathcal{M}$ is *a* monotone class containing the ring $\mathcal{R}$, then $\mathcal{S}\,(\mathcal{R}) \subset \mathcal{M}$.

CHAPTER—III

MEASURES AND OUTER MEASURES

Definition 3.1. Let Ω denote the set of all real numbers. We form the set $\Omega^* = \Omega \cup \{-\infty, +\infty\}$, $-\infty$ and $+\infty$ are two elements not in Ω. We define algebraic operations and order relation among $-\infty$, $+\infty$ and the real number x as follows.

$$-\infty < x < +\infty,$$
$$(\pm\infty) + x = x + (\pm\infty) = \pm\infty,$$
$$(\pm\infty) + (\pm\infty) = \pm\infty,$$
$$x(\pm\infty) = (\pm\infty)x = \pm\infty \text{ if } x > 0,$$
$$= 0 \text{ if } x = 0,$$
$$= \mp\infty \text{ if } x < 0,$$
$$(\pm\infty)(\pm\infty) = +\infty,\ (\pm\infty)(\mp\infty) = -\infty,$$
$$x/(\pm\infty) = 0.$$

The symbols $(+\infty) + (-\infty)$ and $(-\infty) + (+\infty)$ are not defined. The system Ω^* thus formed is called the extended real number system. Any real number is called a finite number ; and $+\infty$ and $-\infty$ are called infinite numbers. The symbol $+\infty$ is usually written as ∞.

Let X be a nonempty set. Throughout this chapter we consider only the supsets of X.

Definition 3.2. Let $\mathscr{F}$ be a non-empty class of subsets of X. A function $\mu : \mathscr{F} \to \Omega^*$ is called a set function.

A set function μ defined on the class $\mathscr{F}$ is said to be

(i) additive if $\mu(E \cup F) = \mu(E) + \mu(F)$ for every pair of disjoint sets E, F in $\mathscr{F}$ with $E \cup F \varepsilon \mathscr{F}$ if the *R. H. S.* is defined

(ii) finitely additive if

$\mu(E_1 \cup E_2 \cup \ldots \cup E_n) = \mu(E_1) + \mu(E_2) + \cdots + \mu(E_n)$ for every finite class $\{E_1, E_2, \ldots, E_n\}$ of mutually disjoint sets in $\mathscr{F}$ with $\cup_{i=1}^{n} E_i \in \mathscr{F}$ when the *R.H.S.* is defined.

(iii) Countably additive if

$$\mu\left(\cup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

for every countable class $\{E_1, E_2, E_3, \cdots\}$ of mutually disjoint sets in $\mathscr{F}$ with $\cup_{i=1}^{\infty} E_i \in \mathscr{F}$ when *R.H.S.* is defined.

(iv) monotone increasing if

$$\mu(E) \le \mu(F)$$

for every pair of sets E, F in $\mathscr{F}$ with $E \subset F$ and monotone decreasing if

$$\mu(E) \ge \mu(F).$$

Definition 3.3 (Measure). A set function μ defined on a ring $\mathscr{R}$ is said to be a measure if it is non-negative, countably additive and $\mu(\phi) = 0$.

Let μ be a measure on a ring $\mathscr{R}$. The measure of E is said to be

(i) finite if $\mu(E) < +\infty$.

(ii) σ-finite if there is a sequence $\{E_i\}_{i=1}^{\infty}$ of sets in $\mathscr{R}$ such that

$$E \subset \cup_{i=1}^{\infty} E_i \text{ and } \mu(E_i) < +\infty \; (i = 1,2,3,\ldots).$$

If $\mu(E)$ is finite [σ – finite] for every set E in $\mathscr{R}$, then the measure μ is called finite [σ – finite].

The measure μ defined on the ring $\mathscr{R}$ is said to be complete if the following condition holds.

If $E \in \mathscr{R}$, $F \subset E$ and $\mu(E) = 0$, then $F \in \mathscr{R}$.

It is easy to see that *a* measure is finitely additive.

Theorem 3.1. Let μ be a measure defined on a ring $\mathscr{R}$. Then μ is monotone increasing and subtractive.

Proof : Let A and B be any two sets in $\mathscr{R}$ and $A \subset B$.

We have

$B = A \cup (B \setminus A)$. The sets A and $B \setminus A$ are disjoint. So

$$\mu(B) = \mu(A) + \mu(B \setminus A). \quad \ldots \quad \ldots \quad (3.1)$$

Since $\mu(B \setminus A) \ge 0$, $\mu(B) \ge \mu(A)$. So μ is monotone inereasing. Suppose that $\mu(A)$ is finite. From (3·1) we have

$$\mu(B \setminus A) = \mu(B) - \mu(A).$$

This gives that μ is subtractive.

Theorem 3·2. Let μ be a measure on the ring $\mathscr{R}$. If $E \in \mathscr{R}$ and $\{E_i\}$ is a countable class of sets in $\mathscr{R}$ with $E \subset \cup_i E_i$, Then

$$\mu(E) \le \sum_i \mu(E_i).$$

Proof : Define the sets $A_1, A_2, A_3 \ldots$.as follows.

$A_1 = E_1$, $A_2 = E_2 \setminus E_1$, $A_3 = E_3 \setminus (E_1 \cup E_2)$, ...

Then the sets $A_1, A_2, A_3, \ldots$ are pairwise disjoint and

$\cup_i A_i = \cup_i E_i$. Also $A_i \subset E_i$ $(i = 1, 2, 3, ..)$.

Since $\mathscr{R}$ is a ring, $A_i \in \mathscr{R}$ for each i. Again, since $E \subset \cup_i A_i$ we have

$$E = E \cap (\cup_i A_i) = \cup_i (E \cap A_i).$$

So $\mu(E) = \sum_i \mu(E \cap A_i) \leq \sum_i \mu(A_i) \leq \sum_i \mu(E_i)$.

Theorem 3·3. Let μ be a measure on the ring $\mathcal{R}$. If $E \varepsilon \mathcal{R}$ and $\{E_i\}$ is a countable class of mutually disjoint sets in $\mathcal{R}$ with $\cup_i E_i \subset E$, then

$$\sum_i \mu(E_i) \leq \mu(E).$$

Proof : Let n be any positive integer. Then

$$\cup_{i=1}^{n} E_i \subset E.$$

Since $\mathcal{R}$ is a ring, $\cup_{i=1}^{n} E_i \in \mathcal{R}$.

Again, since μ is monotone increasing,

$$\mu\left(\cup_{i=1}^{n} E_i\right) \leq \mu(E)$$

or $\sum_{i=1}^{n} \mu(E_i) \leq \mu(E)$ .... (3.2).

Since (3.2) holds for every positive integer n we obtain

$$\sum_i \mu(E_i) \leq \mu(E).$$

Theorem 3.4. Let μ be a measure on the ring $\mathcal{R}$. If $\{E_n\}$ is *an* increasing sequence of sets in $\mathcal{R}$ with $\lim E_n \varepsilon \mathcal{R}$, then

$$\mu\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

Proof : We define the sets $A_1, A_2, A_3, \ldots$ as follows.

$A_1 = E_1$, $A_2 = E_2 \setminus E_1$, $A_3 = E_3 \setminus E_2, \ldots.$

Since $\{E_n\}$ is monotone increasing the sets $A_1, A_2, A_3, \ldots$ are pairwise disjoint and

$$\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} E_i = \lim_{n\to\infty} E_n.$$

Since $\mathcal{R}$ is a ring, $A_i \varepsilon \mathcal{R}$ for each i. If $\mu(Er) = +\infty$ for some positive integer r, then $\mu(E_i) = +\infty$ for all $i \geq r$. Since $E_r \subset \lim E_n$, $\mu(\lim E_n) = +\infty$. Hence

$$\lim_{n\to\infty} \mu(E_n) = \mu\left(\lim_{n\to\infty} E_n\right).$$

Next, suppose that $\mu(E_n)$ is finite for every n. Then $\mu(A_n)$ is also finite for every n. $[\because A_n \subset E_n]$. We have

$E_n = E_{n-1} \cup (E_n \setminus E_{n-1}) = E_{n-1} \cup A_n$.
So $\mu(E_n) = \mu(E_{n-1}) + \mu(A_n)$ or $\mu(A_n) = \mu(E_n) - \mu(E_{n-1})$.
Therefore,

$$\mu(\lim E_n) = \mu\left(\cup_{i=1}^{\infty} A_i\right)$$

$$= \sum_{i=1}^{\infty} \mu(A_i)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(A_i)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \{\mu(E_n) - \mu(E_{n-1})\} \quad \text{(We take } Eo = \phi)$$

$$= \lim_{n\to\infty} \mu(E_n).$$

This proves the theorem.

Theorem 3.5. Let μ be a measure on the ring $\mathcal{R}$. If $\{E_n\}$is a decreasing sequence of sets in $\mathcal{R}$ with $\lim E_n \in \mathcal{R}$ and $\mu(E_1)$ is finite, then

$$\lim_{n\to\infty} \mu(E_n) = \mu\left(\lim_{n\to\infty} E_n\right).$$

Proof : We define the sets $A_1, A_2, A_3, \ldots$ as follows.
$A_1 = E_1 \setminus E_2$, $A_2 = E_2 \setminus E_3$, $A_3 = E_3 \setminus E_4, \ldots$. Then $A_n \varepsilon \mathcal{R}$ for each n and $A_i \cap A_j = \phi$ for $i \neq j$.

Write $A_0 = \lim E_n$. Then $A_0 = \cap_{n=1}^{\infty} E_n$ and $E_1 \setminus A_0 = \cup_{i=1}^{\infty} A_i$

So $$\mu(E_1 \setminus A_0) = \sum_{i=1}^{\infty} \mu(A_i).$$

or $$\mu(E_1) - \mu(A_0) = \sum_{i=1}^{\infty} \mu(A_i)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n-1} \mu(A_i)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n-1} \{\mu(E_i) - \mu(E_{i+1})\}$$

$$= \lim_{n\to\infty} \{\mu(E_1) - \mu(E_n)\}$$

$$= \mu(E_1) - \lim_{n\to\infty} \mu(E_n)$$

or $$\mu(\lim E_n) = \lim_{n\to\infty} \mu(E_n).$$

This completes the proof of the theorem.

Theorem 3·6. Let $\mathscr{P}$ be a semiring and μ be a non-negative countably additive set function on $\mathscr{P}$ with $\mu(\phi) = 0$. Then μ can be extended to *a* unique measure on $\mathscr{R}(\mathscr{P})$.

Proof : We define a set function $\overline{\mu}$ on $\mathscr{R}(\mathscr{P})$ as follows.

Let $A \in \mathscr{R}(\mathscr{P})$. Then $A = \cup_{i=1}^{n} A_i$,

where $A_1, A_2, \ldots, A_n$ are pairwise disjoint sets in $\mathscr{P}$. We define

$$\overline{\mu}(A) = \sum_{i=1}^{n} \mu(A_i).$$

The representation of A as the finite union of pairwise disjoint sets in $\mathscr{P}$ is not unique. Let

$$A = \cup_{i=1}^{m} A_i \text{ and } A = \cup_{i=1}^{n} B_j$$

be two representations of A of above type.
We have

$$A_i = A_i \cap A = A_i \cap \left(\cup_{j=1}^{n} B_j\right) = \cup_{j=1}^{n}\left(A_i \cap B_j\right).$$

So $$\mu(A_i) = \sum_{j=1}^{n} \mu(A_i \cap B_i)$$

and $$\sum_{i=1}^{m} \mu(A_i) = \sum_{i=1}^{m} \sum_{j=1}^{n} \mu(A_i \cap B_i) \qquad \ldots \quad \ldots \quad (3.3)$$

Similarly

$$\sum_{j=1}^{n} \mu(B_i) = \sum_{i=1}^{m} \sum_{j=1}^{n} \mu(A_i \cap B_i) \qquad \ldots \quad \ldots \quad (3.4)$$

From (3·3) and (3·4) we get

$$\sum_{i=1}^{m}\mu(A_i)=\sum_{j=1}^{n}\mu(Bj).$$

This gives that $\bar{\mu}$ (A) is independent of the representation of A. We now complete the proof by the following steps.

(I) Let A and B be any two sets in $\mathcal{R}(\mathcal{P})$ with $A \cap B = \phi$. Then

$A=\cup_{i=1}^{m} A_i$ and $B=\cup_{j=1}^{n} B_j$,

where A_i, B_j ε $\mathcal{P}$ and $A_i \cap A_k = \phi$, $B_i \cap B_k = \phi$ for $i \neq k$.
Write $E_i = A_i$ (i= 1, 2, ..., m), $E_{m+j}=B_j$ (j =1, 2, ... n).

Then $A \cup B = \cup_{i=1}^{m+n} E_i$

and the sets $E_1, E_2, \ldots, E_{m+n}$ are paiswise disjoint. So

$$\bar{\mu}(A\cup B)=\sum_{i=1}^{m+n}\mu(E_i)$$

$$=\sum_{i=1}^{m}\mu(A_i)+\sum_{j=1}^{n}\mu(B_j)$$

$$=\bar{\mu}(t)+\bar{\mu}(B).$$

(II) Let $\{A_n\}$ be a sequence of pairwise disjoint sets in $\mathcal{R}(\mathcal{P})$ with

$$\bigcup_n^{\infty} A_n \in \mathcal{R}(\mathcal{P}).$$

Write $E=\cup_{n=1}^{\infty} A_n$.

(a) First suppose that E ε $\mathcal{P}$.

Let $A_i=\cup_{j=1}^{n_i} A_{ij}$, where

Aij are pairwise disjoint sets in $\mathcal{P}$.

We have

$$\bar{\mu}(A_i)=\sum_{j=1}^{n_i}\mu(A_{i_j}).$$

So
$$\sum_{i=1}^{\infty}\bar{\mu}(A_i)=\sum_{i=1}^{\infty}\sum_{j=1}^{n_i}\mu(A_{ij})=\mu(E).$$

(b). Now suppose that $E \notin \mathcal{P}$.

Then $E = \cup_{j=1}^{n} E_j$, where $E_1, E_2, \ldots E_n$ are pairwise disjoint sets in $\mathcal{P}$.
We have

$$E_j = E \cap E_j = \cup_{i=1}^{\infty} \left(A_i \cap E_j\right).$$

and $A_i = A_i \cap E = \cup_{j=1}^{n} \left(A_i \cap E_j\right).$

By case I.

$$\bar{\mu}(A_i) = \sum_{j=1}^{n} \bar{\mu}\left(A_i \cap E_j\right). \qquad \text{... ...} \quad (3.5)$$

By case (a),

$$\mu\left(E_j\right) = \sum_{i=1}^{\infty} \bar{\mu}\left(A_i \cap E_j\right).$$

So $\bar{\mu}(E) = \sum_{j=1}^{n} \mu\left(E_j\right)$

$$= \sum_{j=1}^{n} \sum_{i=1}^{\infty} \bar{\mu}\left(A_i \cap E_j\right)$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{n} \bar{\mu}\left(A_i \cap E_j\right)$$

$$= \sum_{i=1}^{\infty} \bar{\mu}(A_i) \qquad \text{[By (3·5)].}$$

(III) Let μ_1 and μ_2 be two measures on $\mathscr{P}$ which coincides with μ on $\mathscr{P}$. Let $A \in \mathscr{R}(\mathscr{P})$. Then $A = \cup_{i=1}^{n} A_i$, where $A_1, A_2, \ldots, A_n$ are pairwise disjoint sets in $\mathscr{P}$.

We have

$$\mu_1(A) = \sum_{i=1}^{n} \mu_1(A_i) = \sum_{i=1}^{n} \mu(A_i).$$

$$\mu_2(A) = \sum_{i=1}^{n} \mu_2(A_i) = \sum_{i=1}^{n} \mu(A_i).$$

So $\mu_1(A) = \mu_2(A)$.

This proves the uniqueness of $\bar{\mu}$ and the proof of the theorem is complete.

Outer Measures.

Definition 3.4. A non-empty class $\mathcal{E}$ of sets is said to be hereditary if the conditions $E \varepsilon \mathcal{E}$ and $F \subset E$ imply that $F \varepsilon \mathcal{E}$.

The hereditary σ- ring generated by the class $\mathcal{E}$ is denoted by $\mathcal{H}(\mathcal{E})$ .It is easy to see that $\mathcal{H}(\mathcal{E})$ is the class of all sets which can be covered by countably many sets in $\mathcal{E}$.

Definition 3.5. Let $\mathcal{E}$ be a class of sets and σ be *a* set function on $\mathcal{E}$. The set function σ is said to be

(i) subadditive if for A, B in $\mathcal{E}$ with $A \cup B \in \mathcal{E}$
$\sigma(A \cup B) \leq \sigma(A) + \sigma(B)$ when *R. H. S.* is defined.
(ii) finitely subadditive if for any finite number of sets $A_1, A_2, \ldots, A_n$

with $\bigcup_{i=1}^{n} A_i \in \mathcal{E}$

$\sigma(A_1 \cup A_2 \cup \cdots \cup A_n \leq \sigma(A_1)+\sigma(A_2)+\cdots+\sigma(A_n)$
provided the *R.H.S.* is defined.
(iii) countably subadditive if for any sequence $\{A_i\}$ of sets in $\mathcal{E}$ with $\cup_i A_i \in \mathcal{E}$

$$\sigma\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \sigma(A_i)$$

provided the *R. H. S.* is defined.

Definition 3.6. A set function μ^* defined on the hereditary σ–ring $\mathcal{H}$ is said to be an outer measure if it is non-negative, monotone increasing, countably subadditive and $\mu^*(\phi) = 0$.

Note 3.1. It is easy to see that every outer measure is finitely subadditive.

Theorem 3.7. Let μ be a measure on a ring $\mathcal{R}$ and for every set A in $\mathcal{H}(\mathcal{R})$, let

$$\mu^*(A) = \inf\left\{\sum_{i=1}^{\infty} \mu(A_i) : \ A_i \in \mathcal{R} \text{ and } A \subset \bigcup_{i=1}^{\infty} A_i\right\}.$$

Then μ^* is an outer measure on $\mathcal{H}(\mathcal{R})$ and it is an extension of μ .If μ is σ–finite, then so is μ^*.

Proof : We prove the theorem by the following steps.
(I) Let $E \varepsilon \mathcal{R}$. Write $A_1 = E$, $A_i = \phi$ $(i = 2, 3, \cdots)$. Then $A_i \varepsilon \mathcal{R}$ and

$E \subset \bigcup_{i=1}^{\infty} A_i$. So

$$\mu^*(E) \leq \sum_{i=1}^{\infty} \mu(A_i) = \mu(E) \qquad \ldots \qquad \ldots \qquad (3.6)$$

Again, let $\{A_i\}$ be any sequence of sets in .. /ith $E \subset \bigcup_{i=1}^{\infty} A_i$. Then by theorem 3.2,

$$\mu(E) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

This gives that

$$\mu(E) \leq \mu^{*}(E) \quad \cdots \cdots \cdots (3.7)$$

From (3.6) and (3.7) we get

$$\mu^*(E) = \mu(E).$$

Thus μ^* is an extension of μ .

(II) Let A and B be any two sets in $\mathcal{H}(\mathcal{R})$ with $A \subset B$.

Choose any $\varepsilon > 0$. Then there is a sequence $\{A_i\}$ of sets in $\mathcal{R}$ such that $B \subset \bigcup_{i=1}^{\infty} A_i$ and

$$\sum_{i=1}^{\infty} \mu(A_i) \leq \mu^*(B) + \varepsilon \quad \cdots \cdots (3.8)$$

Since $A \subset B$, $A \subset \bigcup_{i=1}^{\infty} A_i$ .This gives that

$$\mu^*(A) \leq \sum_{i=1}^{\infty} \mu(A_i) \quad \cdots \cdots (3.9)$$

From (3.8) and (3.9) we have

$$\mu^*(A) \leq \mu^*(B) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we obtain

$$\mu^*(A) \leq \mu^*(B).$$

So $\mu^*$is monotone increasing.

(III) Let $\{A_i\}$ be any sequence of sets in $\mathcal{H}(\mathcal{R})$. Suppose that the series $\sum_{i=1}^{\infty} \mu^*(A_i)$ is convergent.

Choose any $\varepsilon > 0$. For each i, there is a sequence $\{A_{ij}\}$ of sets in $\mathcal{R}$ such that

$$A_i \subset \bigcup_{j=1}^{\infty} A_{i_j} \quad \text{and} \quad \sum_{j=1}^{\infty} \mu(A_{i_j}) < \mu^*(A_i) + \frac{\varepsilon}{2^i}.$$

Clearly $\{A_{i_j}\}$ covers the set $\bigcup_{i=1}^{\infty} A_i$.

So

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i,j} \mu\left(A_{i_j}\right) \le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mu\left(A_{i_j}\right)$$

$$\le \sum_{i=1}^{\infty}\left\{\mu^*(A_i) + \varepsilon/2^i\right\} = \sum_{i=1}^{\infty}\mu(A_i) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty}\mu^*(A_i). \qquad \cdots\cdots(3.10)$$

If the series $\sum \mu^*(A_i)$, diverges then clearly (3.10) holds. Hence μ^* is countably subadditive.

From (I) , (II) and (III) we see that $\mu^*$is an outer measure on $\mathcal{H}(\mathcal{R})$ and it is an extension of μ.

(IV). Now suppose that μ is σ– finite. Let E be any set in $\mathcal{H}(\mathcal{R})$. There is a sequence $\{A_i\}$ of sets in $\mathcal{R}$ such that

$$E \subset \bigcup_{i=1}^{\infty} A_i \cdot$$

For each i, we can choose a sequence $\left\{A_{i_j}\right\}$ in $\mathcal{R}$ such that

$A_i \subset \bigcup_{j=1} A_{i_j}$ and $\mu\left(A_{i_j}\right) < +\infty$ (j=1, 2,3, $\cdots$).

Clearly $\left\{A_{i_j}\right\}$ covers the set E and

$\mu\left(A_{i_j}\right) < +\infty$ for i, j =1, 2, 3,$\cdots$

Therefore μ^* is σ–finite.

Definition 3.7. Let μ^* be an outer measure on the hereditary σ–ring $\mathcal{H}$. A set E in $\mathcal{H}$ is said to be μ^*–measurable if for every set A in $\mathcal{H}$

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E'),$$

where E' is the complement of E.

Theorem 3.8. Let μ^* be an outer measure on the hereditary σ–ring $\mathcal{H}$ and $\bar{\mathcal{S}}$ denote the class of all μ^*–measurable sets in $\mathcal{H}$. Then $\bar{\mathcal{S}}$ is *a* ring.

Proof : From definition it follows that the void set ϕ is μ^*–measurable. So $\phi \in \bar{\mathcal{S}}$.

Let E and F be any two sets in $\bar{\mathcal{S}}$.

Take any set A in $\mathcal{H}$. Then

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E') \quad \cdots \quad (3\cdot 11\, a)$$

and $\mu^*(A) = \mu^*(A \cap F) + \mu^*(A \cap F')$ $\quad \cdots \quad (3\cdot 11\, \bar{b})$.

Replacing A by $A \cap E$ and $A \cap E'$ in $(3\cdot 11\bar{b})$ we get

$\mu^*(A \cap E) = \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F')$

and $\mu^*(A \cap E') = \mu^*(A \cap E' \cap F) + \mu^*(A \cap E' \cap F')$.

Now substituting the values of $\mu^*(A \cap E)$ and $\mu^*(A \cap E')$ in (3.11a) we have

$$\mu^*(A) = \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F') + \mu^*(A \cap E' \cap F) + \mu^*(A \cap E' \cap F') \quad \cdots (3\cdot 12)$$

Replacing A by $A \cap (E \cap F)$ in $(3\cdot 12)$

$$\mu^*(A \cap (E \cup F)) = \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F') + \mu^*(A \cap E' \cap F) \quad \cdots (3\cdot 13)$$

From $(3\cdot 12)$ and $(3\cdot 13)$ we have

$\mu^*(A) = \mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)')$.

This gives that $E \cup F$ is μ^* measurable and so $E \cup F \in \bar{\mathcal{S}}$.

Again, replacing A by $A \cap (E/F)' = A \cap (E' \cup F)$ in $(3\cdot 12)$ we get

$$\mu^*(A \cap (E' \cup F)) = \mu^*(A \cap E \cap F) + \mu^*(A \cap E' \cap F) + \mu^*(A \cap E' \cap F') \quad \cdots \quad \cdots \quad (3\cdot 14)$$

From $(3\cdot 12)$ and $(3\cdot 14)$ we obtain

$\mu^*(A) = \mu^*(A \cap (E' \cup F)) + \mu^*(A \cap E \cap F')$

$= \mu^*(A \cap (E \setminus F)) + \mu^*(A \cap (E \setminus F)')$.

This gives that $E \setminus F$ is μ^*–measurable and so $E \setminus F \in \bar{\mathcal{S}}$.

Therefore $\bar{\mathcal{S}}$ is a ring.

Theorem 3.9. Let μ^* be an outer measure on the hereditary σ–ring $\mathcal{H}$ and $\bar{\mathcal{S}}$ denote the class of all measurable sets in $\mathcal{H}$. If $\{E_i\}$ is any sequence of pairwise disjoint sets in $\bar{\mathcal{S}}$ and A is any set in $\mathcal{H}$, then

$$\mu^*\left(A \cap \left(\cup_{i=1}^{\infty} E_i\right)\right) = \sum_{i=1}^{\infty} \mu^*(A \cap E_i).$$

Proof : Let $\{E_i\}$ be any sequence of pairwise disjoint sets in $\bar{\mathcal{S}}$ and A be any set in $\mathcal{H}$.

Since E_1 is μ^*-measurable,

$$\mu^*(A \cap (E_1 \cup E_2)) = \mu^*(A \cap (E_1 \cup E_2) \cap E_1) + \mu^*(A \cap (E_1 \cup E_2) \cap E_1')$$
$$= \mu^*(A \cap E_1) + \mu^*(A \cap E_2) \quad [\because E_1 \cap E_2 = \phi]$$

Now using the method of Induction we obtain

$$\mu^*\left(A\cap\left(\bigcup_{i=1}^{n}E_i\right)\right)=\sum_{i=1}^{n}\mu^*(A\cap E_i) \quad \dots \quad \dots \quad (3.15)$$

for every positive integer n.

Write $E=\bigcup_{i=1}^{\infty}E_i$ and $F_n=\bigcup_{i=1}^{n}E_i$.

Then $F_n\in\bar{\mathcal{S}}$ for every positive integer n. For any n,

$\mu^*(A)=\mu^*(A\cap F_n)+\mu^*(A\cap F_n')$

$$=\sum_{i=1}^{n}\mu^*(A\cap E_i)+\mu^*(A\cap F'_n) \text{ [By (3·15)]}$$

Since $F_n\subset E$, $E'\subset F_n'$ and so

$$\mu^*(A)\geq\sum_{i=1}^{n}\mu^*(A\cap E_i)+\mu^*(A\cap E').$$

This gives that $\mu^*(A)\geq\sum_{i=1}^{\infty}\mu^*(A\cap E_i)+\mu^*(A\cap E') \quad \dots \quad \dots \quad (3·16)$

Replacing A by $A\cap E$ in (3·16) we have

$$\mu^*(A\cap E)\geq\sum_{i=1}^{\infty}\mu^*(A\cap E_i) \quad \dots \quad \dots \quad (3·17)$$

Again,

$$A\cap E=\bigcup_{i=1}^{\infty}(A\cap E_i).$$

So $\mu^*(A\cap E)\leq\sum_{i=1}^{\infty}\mu^*(A\cap E_i) \quad \dots \quad \dots \quad (3·18)$

From (3·17) and (3·18) we get

$$\mu^*(A\cap E)=\sum_{i=1}^{\infty}\mu^*(A\cap E_i).$$

Theorem 3.10. Let μ^* be an outer measure on the hereditary σ-ring $\mathcal{H}$ and $\bar{\mathcal{S}}$ denote the class of all μ^*– measurable sets in $\mathcal{H}$. Then $\bar{\mathcal{S}}$ is *a* σ–ring.

Proof : By Theorem 3.8, $\bar{\mathcal{S}}$ is *a* ring. To show that $\bar{\mathcal{S}}$ is *a* σ–ring we proceed as follows.

(I) Let $\{E_i\}$ be *a* countable class of pairwise disjoint sets in $\bar{\mathcal{S}}$.

Write $E=\bigcup_{i=1}^{\infty}E_i$ and $F_n=\bigcup_{i=1}^{n}E_i$ ($n=1, 2, \dots$).

Then $F_n\in\bar{\mathcal{S}}$ for every positive integer n.

Take any set A *in* $\mathscr{H}$. By Theorem 3.9,

$$\mu^*(A \cap E) = \sum_{i=1}^{\infty} \mu^*(A \cap E_i)$$

and $\mu^*(A \cap F_n) = \sum_{i=1}^{n} \mu^*(A \cap E_i)$.

Again, since $F_n \in \bar{\mathscr{S}}$,

$$\mu^*(A) = \mu^*(A \cap F_n) + \mu^*(A \cap F_n')$$

$$\geq \sum_{i=1}^{n} \mu^*(A \cap E_i) + \mu^*(A \cap E') \qquad (\because E' \subset F_n')$$

This is true for every n. So we get

$$\mu^*(A) \geq \sum_{i=1}^{\infty} \mu^*(A \cap E_i) + \mu^*(A \cap E')$$

or $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E')$ ... ... (3·10)

Since $A = (A \cap E) \cup (A \cap E')$,

$\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E')$ ... ... (3'11)

From (3.10) and (3.11) we get

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E').$$

This gives that E is μ^*– measurable and so $E \in \bar{\mathscr{S}}$.

(II) Now let $\{E_i\}$ be any countable class of in $\bar{\mathscr{S}}$ and let $E = \bigcup_{i=1}^{\infty} E_i$.

Define the sets $F_1, F_2, F_3, \ldots\ldots$ as follows. $F_1 = E_1$ and $F_n = E_n | (E_1 \cup E_2 \cup \ldots \cup E_{n-1})$ $(n \geq 2)$.

Then $F_1\ F_2, F_3, \ldots$ are pairwise disjoint and $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i = E$.

By case *I*, $E \in \bar{\mathscr{S}}$.

Therefore, $\bar{\mathscr{S}}$ is *a* σ–ring.

Theorem 3.11. Let μ^* be an outer measure on the hereditary σ–ring $\mathscr{H}$ and $\bar{\mathscr{S}}$ denote the class of all μ^*– measurable sets in $\mathscr{H}$. Let the set functin $\bar{\mu}$ be defined on $\bar{\mathscr{S}}$ by $\bar{\mu}(E) = \mu^*(E)$ for $E \in \bar{\mathscr{S}}$. Then $\bar{\mu}$ is a complete measure on $\bar{\mathscr{S}}$.

Proof : Clearly $\bar{\mu}$ is non-negative and $\bar{\mu}(\phi) = O$.

Let $\{E_i\}$ be any countable class of pairwise disjoint sets in $\bar{\mathscr{S}}$ and $E = \bigcup_{i=1}^{\infty} E_i$ Then $E \in \bar{\mathscr{S}}$. Take any set A in $\mathscr{H}$. By Theorem 3.9,

$$\mu^*(A \cap E) = \sum_{i=1}^{\infty} \mu^*(A \cap E_i).$$

Taking $A = E$ we get

$$\mu^*(E) = \sum_{i=1}^{\infty} \mu^*(E_i)$$

$$\text{or } \bar{\mu}(E) = \sum_{i=1}^{\infty} \bar{\mu}(E_i).$$

So $\bar{\mu}$ is *a* measure on $\bar{\mathscr{S}}$. Let E be any set in $\bar{\mathscr{S}}$ with $\bar{\mu}(E) = 0$. Take any F in $\mathscr{H}$ with $F \subset E$.

We have

$$\mu^*(F) \le \mu^*(E) = \bar{\mu}(E) = 0.$$

Let A be any set in $\mathscr{H}$. Then

$$\mu^*(A) = \mu^*(F) + \mu^*(A)$$
$$\ge \mu^*(A \cap F) + \mu^*(A \cap F').$$

This gives that F is μ^*–measurable and so $F \in \bar{\mathscr{S}}$.

Therefore, the measure $\bar{\mu}$ is complete.

Theorem 3.12. Let μ be *a* measure on the ring $\mathscr{R}$ and let the outer measure μ^* on $\mathscr{H}(\mathscr{R})$ be defined as follows.

For A in $\mathscr{H}(\mathscr{R})$,

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \mu(E_i) : E_i \in \mathscr{R} \text{ and } A \subset \bigcup_{i=1}^{\infty} E_i \right\}.$$

Then every set in $\mathscr{S}(\mathscr{R})$ is μ^*– measusable.

Proof : Let $\bar{\mathscr{S}}$ denote the class of all μ^*–measurable sets in $\mathscr{H}(\mathscr{R})$. Then $\bar{\mathscr{S}}$ is *a* σ–ring. We show that $\mathscr{R} \subset \bar{\mathscr{S}}$ which gives that $\mathscr{S}(\mathscr{R}) \subset \bar{\mathscr{S}}$.

Take any set E in $\mathscr{R}$. Let A be any set in $\mathscr{H}(\mathscr{R})$. Suppose that $\mu^*(A)$ is finite.

Choose any $\varepsilon > 0$. There is *a* sequence $\{E_i\}$ of sets in $\mathscr{R}$ such that

$$A \subset \bigcup_{i=1}^{\infty} E_i \text{ and } \sum_{i=1}^{\infty} \mu(E_i) < \mu^*(A) + \varepsilon.$$

Now $E_i = (E_i \cap E) \cup (E_i \setminus E)$ and $E_i \cap E$, $E_i \setminus E$ belong to $\mathscr{R}$. So

$$\mu^*(E_i) = \mu(E_i \cap E) + \mu(E_i \setminus E).$$

We have

$$A \cap E \subset \bigcup_{i=1}^{\infty} (E_i \cap E) \text{ and } A \cap E' \subset \bigcup_{i=1}^{\infty} (E_i \cap E').$$

So $\mu^*(A \cap E) + \mu^*(A \cap E')$

$$\leq \sum_{i=1}^{\infty} \{\mu(E_i \cap E) + \mu(E_i \cap E')\}$$

$$= \sum_{i=1}^{\infty} \mu(E_i)$$

$$< \mu^*(A) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we obtain

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E') \quad \dots \quad (3.20)$$

If $\mu^*(A) = +\infty$, then clearly (3.20) holds. Thus (3.20) holds in any case. Hence the set E is measurable. This gives that $\mathcal{R} \subset \bar{\mathcal{S}}$. Since $\bar{\mathcal{S}}$ is *a* σ–ring we have $\mathcal{S}(\mathcal{R}) \subset \bar{\mathcal{S}}$. This proves the theorem.

Theorem 3.13. Let μ be *a* σ– finite measure on the ring $\mathcal{R}$, then μ can be extended to *a* unique σ–finite measure $\bar{\mu}$ on $\mathcal{S}(\mathcal{R})$.

Proof : Let us define the set function μ^* on $\mathcal{H}(\mathcal{R})$ as follows.

For any set A in $\mathcal{H}(\mathcal{R})$,

$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \mu(E_i) : E_i \in \mathcal{R} \text{ and } A \subset \bigcup_{i=1}^{\infty} E_i \right\}$. Then μ^* is an outer measure on $\mathcal{H}(\mathcal{R})$ and $\mu^*(E) = \mu(E)$ for all $E \in \mathcal{R}$.

Denote by $\bar{\mathcal{S}}$ the class of all μ^*–measurable sets in $\mathcal{H}(\mathcal{R})$. Then $\bar{\mathcal{S}}$ is *a* σ–ring and $\mathcal{R} \subset \bar{\mathcal{S}}$ which gives that $\mathcal{S}(\mathcal{R}) \subset \bar{\mathcal{S}}$.

Now define $\bar{\mu}(E) = \mu^*(E)$ for $E \in \bar{\mathcal{S}}$. Then $\bar{\mu}$ is *a* measure on $\bar{\mathcal{S}}$ and so *a* measure on $\mathcal{S}(\mathcal{R})$; and also $\bar{\mu}$ is σ–finite.

Let μ_1 and μ_2 be two measures on $\mathcal{S}(\mathcal{R})$ such that $\mu_1(E) = \mu_2(E) = \mu(E)$ for all E in $\mathcal{R}$.

(I) Let A be any set in $\mathcal{R}$ with $\mu(A) < +\infty$. Denote by $\mathcal{M}$ the class of all sets E in $\mathcal{S}(\mathcal{R}) \cap A = \mathcal{S}(\mathcal{R} \cap A)$ such that $\mu_1(E) = \mu_2(E)$. Then clearly $\mathcal{R} \cap A \subset \mathcal{M}$. Let $\{E_i\}$ be any monotone sequence of sets in $\mathcal{M}$. Write $E = \lim E_i$. Since $E_n \subset A$, $\mu_i(E_n) \leq \mu_i(A) = \mu(A) < +\infty$. So $\lim \mu_i(\lim E_n) = \mu_i(E)$. Now $\mu_1(E_n) = \mu_2(E_n)$ for all n which gives that $\mu_1(E) = \mu_2(E)$. So $E \in \mathcal{M}$. Hence $\mathcal{M}$ is *a* monotane class. This gives that $\mathcal{S}(\mathcal{R} \cap A) \subset \mathcal{M}$. But from definition $\mathcal{M} \subset \mathcal{S}(\mathcal{R} \cap A)$. So $\mathcal{M} = \mathcal{S}(\mathcal{R} \cap A)$.

(II) Let E be any set in $\mathcal{S}(R)$. Then there exists *a* sequence $\{A_i\}$ of pairwise disjoint sets in $\mathcal{R}$ such that

$$E \subset \bigcup_{i=1}^{\infty} A_i \text{ and } \mu(A_i) < +\infty \ (i = 1,2,3,\dots.)$$

We have

So $$E = \bigcup_{i=1}^{\infty} (E \cap A_i)$$

$$\mu_1(E) = \sum_{i=1}^{\infty} \mu_1 (E \cap A_i)$$

$$\text{and } \mu_2(E) = \sum_{i=1}^{\infty} \mu_2(E \cap A_i) :$$

Since $E \cap A_i \in \mathcal{S}(R \cap A_i)$ and $\mu (A_i) < +\infty$, by case I,

$$\mu_1(E \cap A_i) = \mu_2(E \cap A_i) \; (i = 1,2,3, \ldots).$$

So $$\mu_1 (E) = \mu_2 (E).$$

Hence $\mu_1 = \mu_2$ on $\mathcal{S}(\mathcal{R})$.

This completes the proof of the theorem.

Lebesgue Measure on the Real Line.

Let $\mathcal{P}$ denote the class of all intervals of the form $[a,b)$, where a and b are real numbers with $a \leq b$. Then $\mathcal{P}$ is a semiring (See Example 2.3, Ch-II). We define the set function μ on $\mathcal{P}$ as follows.

Let $I = [a, b)$ belong to $\mathcal{P}$. Then $\mu(I) = b - a$. Clearly μ is non-negative. If $a = b$, then $I = \phi$ and so $\mu(\phi) = 0$. Also μ is increasing.

Theprem 3.14. The set function μ defined on $\mathcal{P}$ is countably additive.

Proof : Let $\{E_i\}$ be a sequence of pairwise disjoint sets in $\mathcal{P}$ with

$$E = \bigcup_{i=1}^{\infty} E_i \in \mathcal{P}.$$

Write $E = [a, b)$ and $E_i = [a_i, b_i)$ $(i = 1,2,3, \ldots)$.

Let n be a positive integer. We consider the sets $E_1, E_2, E_3, \ldots; E_n$. Without loss of generality we may suppose that $a \leq a_1 < b_1 \leq a_2 < b_2 \leq \ldots\ldots \leq a_n < b_n \leq b$. Then

$$\sum_{i=1}^{n} (b_i - a_i) \leq \sum_{i=1}^{n-1} (a_{i+1} - a_i) + (b - a_n) = b - a_1 \leq b - a.$$

This gives that

$$\sum_{i=1}^{n} \mu (E_i) \leq \mu (E).$$

Since n is arbitrary we get

$$\sum_{i=1}^{\infty} \mu (E_i) \leq \mu(E) \qquad \ldots \qquad \ldots \qquad (3.21)$$

To show the reverse inequality we proced as follows.
Choose any ε with $0 < \varepsilon < b - a$.

Write $I_i = \left(a_i - \varepsilon/2^i, b_i\right)$ $(i = 1, 2, 3, ...)$ and $\Delta = \{I_i : i = 1, 2, 3,....\}$. Clearly Δ covers the closed interval $[a, b - \varepsilon]$. By Heine-Borel Theorem we can find a positive integer m such that

$$[a, b - \varepsilon] \subset \bigcup_{i=1}^{m} I_i$$

Then

$$b - a - \varepsilon < \sum_{i=1}^{m} |I_i| < \sum_{i=1}^{\infty} |I_i| = \sum_{i=1}^{\infty} (b_i - a_i) + \varepsilon.$$

or $\mu(E) < \sum_{i=1}^{\infty} \mu(E_i) + 2\varepsilon$

Since $\varepsilon > 0$ is arbitrary we obtain

$$\mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i) \qquad .... \qquad ... \qquad (3{\cdot}22)$$

From (3.21) and (3.22) we get

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i).$$

Hence the set function μ defined on $\mathcal{P}$ is countably additive.

Lebesgue Outer Measure.

Let $\mathcal{R} = \mathcal{R}(\mathcal{P})$ be the ring generated by the semiring $\mathcal{P}$. By Theorem 3.6, there exists a unique extension of μ to a measure on the ring $\mathcal{R}$. We also denote this extension by μ.

Let $\mathcal{H} = \mathcal{H}(\mathcal{R})$ be the hereditaty σ–ring generated by the ring $\mathcal{R}$. We can verify that $\mathcal{H} = \mathcal{H}(\mathcal{P})$. Clearly $\mathcal{H}$ consists of all subsets of the real line. Define the set function μ^* on $\mathcal{H}$ as fllows.

Let $E \in \mathcal{H}$. Then

$$\mu^*(E) = \inf\left\{\sum_{i=1}^{\infty} \mu(E_i)\,;\ E_i \in \mathcal{R} \text{ and } E \subset \bigcup_{i=1}^{\infty} E_i\right\}.$$

By Theorem 3.7, μ^* is an outer measare on $\mathcal{H}$ and $\mu^*(E) = \mu(E)$ for all $E \in \mathcal{R}$.

The outer measure μ^* thus defined is called Lebesgue outer measure on the real line.

Let $\{E_i\}$ be any sequence of sets in $\mathscr{R}$. For each i, we can write

$$E_i = \bigcup_{j=1}^{n_i} I_{ij}$$

where $I_{ij} \in \mathscr{P}$ and $I_{ij} \cap I_{ik}$ for $j \neq k$. Also $\mu(E_i) = \sum_{j=1}^{n_i} \mu(I_{ij})$.

So
$$\sum_{i=1}^{\infty} \mu(E_i) = \sum_{i=1}^{\infty} \sum_{j=1}^{n_i} \mu(I_{ij}).$$

We can arrange the sets

$$I_{ij}\ (j=1,2,\ldots, n_i : i = 1, 2, 3, \ldots..)$$

in the form of a sequence as

$$I_1, I_2, I_3, \ldots.., I_n, \ldots.$$

Then
$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} I_i$$

and
$$\sum_{i=1}^{\infty} \mu(E_i) = \sum_{i=1}^{\infty} \mu(I_i).$$

From it follows that for $E \in \mathscr{H}$,

$$\mu^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu(E_i) : E_i \in \mathscr{P} \text{ and } E \subset \bigcup_{i=1}^{\infty} E_i \right\}$$

Lebesgue Measure and Lebesgue Measurable sets.

Denote by $\bar{\mathscr{S}}$ the class of all μ^*–measurable sets in the hereditary σ–ring $\mathscr{H}$ and let $\bar{\mu}(E) = \mu^*(E)$ for $E \in \bar{\mathscr{S}}$. Then by Theorems 3.10, 3.11 and 3.12 we see that $\bar{\mathscr{S}}$ is a σ–ring, $\bar{\mu}$ is a complete measure on $\bar{\mathscr{S}}$ and $\mathscr{S}(\mathscr{R}) \subset \bar{\mathscr{S}}$. Further $\bar{\mu}$ is σ–finite.

The measure $\bar{\mu}$ is called the Lebesgue Measure on the real line and a subset E of the real line is said to be Lebesgue Measurable if $E \in \bar{\mathscr{S}}$.

Note 3.1. Let a and b be any two real numbers with $a < b$. Choose any decreasing sequence $\{a_n\}$ in (a, b) with $\lim a_n = a$. Then clearly (a, b) $= \bigcup_{n=1}^{\infty} [a_n, b)$. Also $[a, b] = \bigcap_{n=1}^{\infty} \left[a, b + \frac{1}{n}\right)$. This gives that $\mathscr{S}(\mathscr{R})$ contains all finite open and closed intervals. We can verify that the sets

(a, ∞), $[a, \infty)$, $(-\infty, b)$, $(-\infty, b]$, $(-\infty, \infty)$ all belong to $\mathscr{S}(\mathscr{R})$.

So all open and closed sets belong to $\mathscr{S}(\mathscr{R})$. Hence open and closed set are Lebesgue measurable.

Theorem 3.15. Let $E \in \bar{\mathscr{S}}$ and $\bar{\mu}(E) < +\infty$. Then for any $\varepsilon > 0$, There is an open set $G \supset E$ and *a* closed set $F \subset E$ such that

$$\bar{\mu}\ (G) < \bar{\mu}\ (E) + \varepsilon \text{ and } \bar{\mu}\ (F) > \bar{\mu}\ (E) - \varepsilon.$$

Proof : (I) Let $E \in \bar{\mathscr{S}}$ and $\bar{\mu}(E) < +\infty$. Choose any $\varepsilon > 0$. There is *a* sequence $\{I_i\}$ of sets in $\mathscr{P}$ such that

$$E \subset \bigcup_{i=1}^{\infty} I_i \text{ and } \sum_{i=1}^{\infty} \bar{\mu}(I_i) < \bar{\mu}(E) + \frac{1}{2}\varepsilon \qquad \ldots \qquad \ldots \qquad (3{\cdot}23)$$

Let $I_i = [a_i, b_i)$ $(i = 1, 2, 3, \ldots.)$.

Write $\Delta_i = \left(a_i - \varepsilon/2^i, b_i\right)$ $(i = 1, 2, 3, \ldots.)$

and $G = \bigcup_{i=1}^{\infty} \Delta_i$.

Then $\bigcup_{i=1}^{\infty} I_i \subset G$ and

$$\bar{\mu}\ (G) \le \sum_{i=1}^{\infty} \bar{\mu}(\Delta_i) = \sum_{i=1}^{\infty} \mu(I_i) + \frac{1}{2}\varepsilon$$

or $\bar{\mu}\ (G) < \bar{\mu}\ (E) + \varepsilon$. [By (3·23)]

(II) Suppose that E is bounded. Choose any $\varepsilon > 0$. Denote by *a* and *b* the glb and lub of the set E. Now choose real number α and β in E with $a < \alpha < \beta < b$ such that

$$\alpha - a < \frac{1}{4}\varepsilon \text{ and } b - \beta < \frac{1}{4}\varepsilon.$$

Write $E_1 = [a, \alpha) \cap E$, $E_2 = [\alpha, \beta] \cap E$, $E_3 = (\beta, b] \cap E$.

Then $E = E_1 \cup E_2 \cup E_3$ and

$$\bar{\mu}\ (E) = \bar{\mu}\ (E_1) + \bar{\mu}\ (E_2) + \bar{\mu}\ (E_3) < \bar{\mu}\ (E_2) + \frac{1}{2}\varepsilon.$$

Let $A = [\alpha, \beta] \setminus E_2 = (\alpha, \beta) \setminus E_2$ $(\because \alpha, \beta \in E_2)$.

By case I, there is an open set $G \subset (\alpha, \beta)$ such that

$$A \subset G \text{ and } \bar{\mu}\ (G) < \bar{\mu}\ (A) + \frac{1}{2}\varepsilon.$$

Let $F = [\alpha,\beta] \setminus G$. Then F is *a* closed set and

$$F \cup G = A \cup E_2.$$

Since $A \subset G$, $F \subset E_2 \subset E$.

So $\quad \bar{\mu}(F) + \bar{\mu}(G) = \bar{\mu}(A) + \bar{\mu}(E_2)$

or $\quad \bar{\mu}(F) = \bar{\mu}(E_2) + \bar{\mu}(A) - \bar{\mu}(G)$

or $\quad \bar{\mu}(F) > \bar{\mu}(E) - \varepsilon.$

(III) Suppose that the set E is unbounded. Choose any $\varepsilon > 0$. For each positive integer n, let

$$E_n = (-n, n) \cap E.$$

Then $E_1 \subset E_2 \subset E_3 \subset \ldots$ and $\lim E_n = E$.

We can find the positive integer N such that

$$\bar{\mu}(E_N) > \bar{\mu}(E) - \frac{1}{2}\varepsilon.$$

By case II, there is a closed set $F \subset E_N \subset E$ such that

$$\bar{\mu}(F) > \bar{\mu}(E_N) - \frac{1}{2}\varepsilon > \bar{\mu}(E) - \varepsilon.$$

This completes the proof of the theorem.

Note 3.2. From above discussion we see that Lebesgue measure is an extension of length of an interval to the class of all measurable subsets of Ω.

Note 3.3. Let E be a subset of the real line Ω and $T : \Omega \to \Omega$ be defined by

$$T(x) = \alpha x + \beta,$$

where α and β are real numbers and $\alpha \neq 0$. Let $T(E) = \{T(x) : x \in E\}$. It is easy to verify that set $T(E)$ is Lebesgue measurable if and only if E is so and $\bar{\mu}(T(E)) = |\alpha| \bar{\mu}(E)$.

Lebesgue-Stielties Measure on the real line.

Let Ω denote the set of all real numbers and let $w : \Omega \to \Omega$ be a non-constant increasing function continuous on the left. Denote by $\mathcal{P}$ the class of all sets of the form $[a, b)$, where a and b are real numbers with $a \leq b$. Then $\mathcal{P}$ is a semiring. We define the set function μ on $\mathcal{P}$ by $\mu([a, b)) = w(b) - w(a)$. Clearly μ is non-negative, increasing and $\mu(\phi) = 0$.

Lemma 3.1. Let $I = [a, b)$ and $I_i = [a_i = b_i)$ $(i = 1, 2)$.

If $I \subset I_1 \cup I_2$, then $\mu(I) \leq \mu(I_1) + \mu(I_2)$.

Proof : Suppose that $I \subset I_1 \cup I_2$. We may take $I_1 \cap I_2 \neq \phi$. Without loss of generality we may take $a_1 < a < a_2 \leq b_1 < b \leq b_2$.

Since w is increasing we have

$$\begin{aligned} w(b) - w(a) &\leq w(b_2) - w(a_1) \\ &= \{w(b_2) - w(a_2)\} + \{w(a_2) - w(a_1)\} \\ &\leq \{w(b_2) - w(a_2)\} + \{w(b_1) - w(a_1)\} \end{aligned}$$

or $\mu(I) \le \mu(I_1) + \mu(I_2)$

Lemma 3.2. Let $I = [a, b)$ and $I_i = [a_i, b_i)$ $(i = 1, 2, ..., n)$.
If $I \subset I_1 \cup I_2 \cup .... \cup I_n$, Then

$$\mu(I) \le \mu(I_1) + \mu(I_2) + \cdots + \mu(I_n).$$

Proof : Suppose that $I \subset I_1 \cup I_2 \cup ... \cup I_n$. Without loss of generality we may suppose that the end points of the intervals $I_1, I_2, ..., I_n$ are in ascending order. Then clearly

$$a_1 \le a < a_2 \le b_1 < a_3 \le b_2 < a_4 \le b_3 < \cdots < a_n \le b_{n-1} < b \le b_n.$$

Since w is increasing we have

$$\sum_{i=1}^{n} \{w(b_i) - w(a_i)\}$$

$$= \sum_{i=1}^{n-1} \{w(b_i) - w(a_i)\} + \{w(b_n) - w(a_n)\}$$

$$\ge \sum_{i=1}^{n-1} \{w(a_{i+1}) - w(a_i)\} + \{w(b_n) - w(a_n)\}$$

$$= \{w(a_n) - w(a_1)\} + \{w(b_n) - w(a_n)\}$$
$$= \{w(b_n) - w(a_1)\}$$
$$\ge w(b) - w(a).$$

$$\text{So } \mu(I) \le \sum_{i=1}^{n} \mu(I_i).$$

Theorem 3.16. The set function μ defined on $\mathscr{P}$ is countably additive.

Proof : Let $\{E_i\}$ be *a* sequence of pairwise disjoint sets in $\mathscr{P}$ with

$$\bigcup_{i=1}^{\infty} E_i = E \in \mathscr{P}.$$

Write $E = [a,b)$ and $E_i = [a_i,b_i)$ $(i = 1, 2, 3, ...)$.

Let n be any positive integer. We consider the sets $E_1, E_2, ..., E_n$. We may suppose that $E_1, E_2, ..., E_n$ are in the order of increasing end points. Since $E_1 \cup E_2 \cup ... \cup E_n \subset E$, we have

$$a \le a_1 < b_1 \le a_2 < b_2 \le .... \le a_n < b_n \le b.$$

So $w(a) \le w(a_1) \le w(b_1) \le w(a_2) \le w(b_2) \le w(a_3) \le ....$
$\le w(a_n) \le w(b_n) \le w(a).$

We have

$$\sum_{i=1}^{n} \mu(E_i) = \sum_{i=1}^{n} \{w(b_i) - w(a_i)\}$$

$$= \sum_{i=1}^{n-1} \{w(b_i) - w(a_i)\} + \{w(b_n) - w(a_n)\}$$

$$\leq \sum_{i=1}^{n-1} \{w(a_{i+1}) - w(a_i)\} + \{w(b_n) - w(a_n)\}$$

$$= \{w(a_n) - w(a_1)\} + \{w(b_n) - w(a_n)\}$$

$$= \{w(b_n) - w(a_1)\} \leq \{w(b) - w(a)\}.$$

So $$\sum_{i=1}^{n} \mu(E_i) \leq \mu(E).$$

This gives that

$$\sum_{i=1}^{\infty} \mu(E_i) \leq \mu(E) \quad \ldots \quad \ldots \quad (3{\cdot}24)$$

To prove the reverse inequality we proceed as follows.
Let ε be any positive number.
Choose points $\beta, \alpha_1, \alpha_2, \alpha_3, \ldots\ldots$
with $a < \beta < b$ and $\alpha_i < a_i$ $(i = 1,2,3, \ldots.)$ such that

$$w(b) - w(\beta) < \varepsilon \text{ and } w(a_i) - w(\alpha_i) < \frac{\varepsilon}{2^i} \; (i = 1,2,3, \ldots.).$$

Write $\Delta_i = (\alpha_i, b_i)$ and $\mathscr{F} = \{\Delta_i : i = 1,2,3,\ldots\}$.

Then $\mathscr{F}$ is an open cover of the closed interval $[a, \beta]$. By Heine-Borel Theorem there is a positive integer n such that

$$[a, \beta] \subset \bigcup_{i=1}^{n} \Delta_i \subset \bigcup_{i=1}^{n} [\alpha_i, \beta_i).$$

So by Lemma 3·2,

$$\mu([a, \beta)) \leq \sum_{i=1}^{n} \mu([\alpha_i, \beta_i))$$

or $$w(\beta) - w(a) \leq \sum_{i=1}^{n} \{w(\beta_i) - w(\alpha_i)\}$$

or $$\{w(b) - w(a) - \{w(b) - w(\beta)\}$$

$$\leq \sum_{i=1}^{n} \{w(b_i) - w(a_i)\} + \sum_{i=1}^{n} \{w(a_i) - w(\alpha_i)\}$$

or $$\mu(E) < \sum_{i=1}^{n} \mu(E_i) + 2\varepsilon < \sum_{i=1}^{\infty} \mu(E_i) + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, weget

$$\mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i) \qquad \ldots \qquad \ldots \qquad (3\cdot25)$$

From (3.24) and (3·25) we obtain

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i).$$

Hence μ is countably additive.

Lebesgue-Stieltjes Outer Measure.

Let $\mathcal{R} = \mathcal{R}(\mathcal{P})$ and $\mathcal{H} = \mathcal{H}(\mathcal{R})$.

By Theorem 3.6 there exists *a* unique measure $\bar{\mu}$ on the ring $\mathcal{R}$ such that $\bar{\mu}(E) = \mu(E)$ for all E in $\mathcal{P}$. We write μ in place of $\bar{\mu}$.

For any set E in $\mathcal{H}$, we define

$$\mu^*(E) = \inf\left\{\sum_{i=1}^{\infty} \mu(E_i) : E_i \in \mathcal{R} \text{ and } E \subset \bigcup_{i=1}^{\infty} E_i\right\}.$$

By Theorem 3·7, μ^* is an outer measare on $\mathcal{H}$ and $\mu^*(E) = \mu(E)$ for all $E \in \mathcal{R}$.

This outer measure μ^* is called the Lebesgue-stielties Outer Measure on Ω generated by the function w.

As in the case of Lebesgue outer measure we see that for E in $\mathcal{H}$,

$$\mu^*(E) = \inf\left\{\sum_{i=1}^{\infty} \mu(I_i) : I_i \in \mathcal{P} \text{ and } E \subset \bigcup_{i=1}^{\infty} I_i\right\}.$$

In this case also $\mathcal{S}(\mathcal{R}) \subset \bar{\mathcal{S}}$, where $\bar{\mathcal{S}}$ is the class of all μ^*– measurable sets in $\mathcal{H}$. If $\bar{\mu}$ is the restriction of μ^* on $\bar{\mathcal{S}}$, then $\bar{\mu}$ is *a* complete σ–finite measure on $\bar{\mathcal{S}}$.

$\bar{\mu}$ is called Lebesque-stieltjes Measure on Ω. *A* subset E of Ω is said to be Lebesgue-stieltjes measurable if $E \in \bar{\mathcal{S}}$.

Theorem 3·17. Let $E \in \bar{\mathcal{S}}$ and $\bar{\mu}(E) < +\infty$. Then for every $\varepsilon > 0$, there is an open set $G \supset E$ and *a* closed set $F \subset E$ such that

$\bar{\mu}(G) < \bar{\mu}(E) + \varepsilon$ and $\bar{\mu}(F) > \bar{\mu}(F) - \varepsilon$.

The Proof is analogous to that of Theorem 3.15.

Theorem 3·18. Let μ denote the Lebesgue measure or Lebesgue-Stieltjes measure on the real line Ω. If A is any subset of Ω, then there is *a* measurable set $E \supset A$ such that $\mu^*(A) = \mu^*(E)$.

Proof : Let A be any subset of Ω. For every positive integer n, there is an open set $G_n \supset A$ such that

$$\mu(G_n) \leq \mu^* (A) + \frac{1}{n}.$$

We may suppose that

$$G_1 \supset G_2 \supset G_3 \supset ....$$

Let $E = \bigcap_{n=1}^{\infty} G_n$. Then E is measurable, $E \supset A$ and

$$\mu(E) \leq \mu^* (A) + \frac{1}{n} (n = 1,2,3,....).$$

This gives that

$$\mu(E) \leq \mu^*(A) \quad ... \quad ... \quad (3.26)$$

Again, $A \subset E$. So

$$\mu^*(A) \leq \mu (E) \quad ... \quad ... \quad (3.27)$$

From (3·26) and (3·27) we get

$$\mu(E) = \mu^*(A).$$

Note 3·4. The set E obtained in Theorem 3·18 is called *a* measurable cover of the set A.

Theorem 3·19. Let μ denote the Lebesgue measure or Lebesgue-Stieltjes measure on the real line Ω. If $\{A_n\}$ is any increasing sequence of sets of real numbers and $A = \bigcup_{n=1}^{\infty} A_n$, Then

$$\lim_{n\to\infty} \mu^*(A_n) = \mu^*(A).$$

Proof : Since $A_n \subset A$ for all n, we have

$$\mu^*(A_n) \leq \mu^* (A) \quad ... \quad ... \quad ... \quad (3.28)$$

There is *a* measurable set $E_n \supset A_n$ such that $\mu(E_n) = \mu^* (A_n)$.

Write $B_n = \bigcap_{i=n}^{\infty} E_i$.

Then each B_n is measurable and

$$B_1 \subset B_2 \subset B_3 \subset .... ; \text{ and } A_n \subset B_n \subset E_n.$$

Let $B = \bigcup_{i=1}^{\infty} B_i$. Then B is measurable.

By Theorem 3.4,

$$\lim_{n\to\infty} \mu(B_n) = \mu(B)$$

Also $\mu^*(A_n) \leq \mu (B_n) \leq \mu(E_n) = \mu^*(A_n)$.

This gives that

$$\mu(B_n) = \mu^*(A_n).$$

Therefore

$$\lim_{n\to\infty} \mu^*(A_n) = \mu(B) \quad \ldots \quad (3.29)$$

Since $B_n \supset A_n$ for all n we have $B \supset A$.

So $\mu(B) \geq \mu^*(A)$. ... ... (3.30)

From (3.28), (3.29) and (3.30) we obtain

$$\lim_{n\to\infty} \mu^*(A_n) = \mu^*(A).$$

Corollary 3.19.1. Let μ be as in Theorem 3.19. If $\{A_n\}$ is an increasing sequence of sets, $A = \bigcup_{n=1}^{\infty} A_n$ and B is any set, then

$$\lim_{n\to\infty} \mu^*(A_n \cap B) = \mu^*(A \cap B).$$

Theorem 3.20. Let μ be *a* measure on the real line Ω such that every finite interval I is μ–measurable and $\mu(I) < +\infty$. Then the measure μ is the Lebesgue-stieltjes measure on Ω generated by an increasing function w continuous on the left such that

$$\mu([a, b)] = w(b) - w(a),$$

where a and b are real numbers with $a \leq b$.

Proof : Let us define the function $w: \Omega \to \Omega$ as follows.

For any real number x,

$$w(x) = \mu([0, x)] \text{ if } x \geq 0,$$
$$= -\mu([x, 0)) \text{ if } x < 0.$$

Clearly $w(x)$ is finite for each $x \in \Omega$ and $w(0) = 0$.

Let a and b be any two real numbers with $a < b$.

(i) Suppose that $0 \leq a < b$.

We have $[0, b) = [0, a) \cup [a, b)$.

So $\mu([0, b)) = \mu([0, a)) + \mu([a, b))$

or $w(b) = w(a) + \mu([a, b))$

or $\mu([a, b)) = \mu(b) - w(a)$.

(ii) Let $a < b \leq 0$.

We have $[a, 0) = [a, b) \cup (b, 0)$

So $\mu([a, 0)) = \mu([a, b)) + \mu([b, 0))$

or $-w(a) = \mu([a, b)) - w(b)$

or $\mu(a, b)) = w(b) - w(a)$.

(iii) Lastly let $a < 0 \le b$.

We have $[a, b) = [a, 0) \cup [0, b)$.

So $\mu([a, b)) = \mu([a, 0)) + \mu([0, b)]$.

$= w(b) - w(a)$.

From above we see that $w(b) \ge w(a)$.

So w is increasing.

Take a strictly increasing sequence $\{a_n\}$ with $\lim\limits_{n\to\infty} a_n = a$.

Write $E = [a, b)$ and $E_n = [a_n, b)$ $(n = 1, 2, 3, \ldots)$.

Then $E_1 \supset E_2 \supset E_3 \supset \cdots$ and $\lim E_n = E$.

So $$\mu(E) = \lim_{n\to\infty} \mu(E_n).$$

or $$\mu(b) - w(a) = \lim_{n\to\infty} \{w(b) - w(a_n)\}$$

or $$w(a) = \lim w(a_n) \quad \ldots \quad \ldots \quad (3.31)$$

Since $w(a-)$ exists $w(a_n) \to w(a-)$.

From (3.31) we have

$$w(a) = w(a-).$$

This completes the proof.

CHAPTER—IV

MEASURABLE FUNCTIONS

Let X be a non-empty set and $\mathcal{S}$ be a σ- ring of subsets of X such that $X \in \mathcal{S}$. Then the pair $(X, \mathcal{S})$ is called a measurable space. A subset E of X is said to be measurable $(\mathcal{S})$ or simply measurable if and only if $E \in \mathcal{S}$.

Let $(X \mathcal{S})$ be a measusable space and μ be a measure on $\mathcal{S}$. Then the triplet $(X, \mathcal{S}, \mu)$ is called a measure space. The measure space $(X, \mathcal{S}, \mu)$ is called finite or σ-finite or complete according as μ is finite or σ-finite or complete.

Definition 4.1. Let $(X, \mathcal{S})$ be a measurable space. A function $f: X \to \Omega^*$ (extended real number system) is said to be measurable $(\mathcal{S})$ or simpley measurable if the set $\{x : f(x) > a\}$is measurable for every real number a.

Let E be a subset of X. A funetion $f: E \to \Omega^*$ is said to be measurable on the set E if E is measurable and the set $E(f > a) = \{x : x \in E \text{ and } f(x) > a\}$ is measurable for every real number a.

Theorem 4.1. Let $E = \cup_k E_k$ (countable union) and let $f: E \to \Omega^*$. If f is measurable on E_k for each k, then f is measurable on E.

Proof : Let a be any real number. We have

$$E(f>a) = \cup_k E_k (f > a).$$

Since f is measurable on E_k, the set E_k as well as the set $E_k (f > a)$ are measurable. So the set E and $E (f > a)$ are measurable. Hence f is measurable on E.

Theorem 4.2. Let E be a measurable set and $f: E \to \Omega^*$. If for every real number a one of the sets

(i) $E (f > a)$ (ii) $E (f \geq a)$ (iii) $E(f<a)$ (iii) $E(f < a)$ (iv) $E(f \leq a)$ is measurable, then so are other three.

Proof : Let a be any real number.

(i) Suppose that the set $E (f > r)$ is measurable for every real number r. Since $E(f \leq a) = E \setminus E (f > a)$, it follows that $E(f \leq a)$ is measurable.

Let $E_n = E(f > a - \frac{1}{n})$ $(n = 1, 2, 3, ...)$. Then each E_n is measurable. We have

$$E (f \geq a) = \bigcap_{n=1}^{\infty} E_n.$$

So $E(f \geq a)$ is measurable.

Since $E(f < a) = E \setminus E (f \geq a)$, the set $E(f < a)$ is measurable.

(ii) Suppose that $E(f \geq \alpha)$ is measurable for every real number α.

Since $E(f < \alpha) = E \setminus E(f \geq \alpha)$, $E(f < \alpha)$ is measurable.

Let $E_n = E(f < a + \frac{1}{n})$ $(n = 1, 2, 3, \ldots)$.

We have $E(f \leq a) = \bigcap_{n=1}^{\infty} E_n$.

So $E(f \leq a)$ is measurable.

Since $E(f > a) = E \setminus E(f \leq a)$, $E(f > a)$ is measuble.

The cases (iii) and (iv) may be similarly treated.

Theorem 4.3. If $f : E \rightarrow \Omega^*$ is measurable on E and k ($\neq 0$) be any real number, the functions (i) $f + k$ (ii) kf (iii) $|f|$ (iv) f^2 are measurable on E.

Proof : Let a be any real number. We have

(i) $E(f + k > a) = E(f > a - k)$,

(ii) $E(kf > a) = E(f > \frac{a}{k})$ if $k > 0$

$= E(f < \frac{a}{k})$ if $k < 0$,

(iii) $E(|f| > a) = E(f > a) \cup E(f < -a)$ if $a \geq 0$,

$= E$ if $a < 0$.

(iv) $E(f^2 > a) = E(f > \sqrt{a}) \cup E(f < -\sqrt{a})$ if $a \geq 0$.

$= E$ if $a < 0$.

From above considerations it follows that the functions $f + k$, kf, $|f|$ and f^2 are measurable on E.

Theorem 4.4. If the functions f and g are measurable on E, then the functions (i) $f - g$ (ii) $f + g$ and (iii) fg are measurable on E.

Proof : We prove the theorem by the following steps.

(I) We enumerate the rational numbers as follows : $r_1, r_2, r_3, \ldots, r_n, \ldots$

Since f and g are measurable on E, the sets $E(f > r_n)$ and $E(g < r_n)$ are measurable.for every n. Hence the set

$$A = \bigcup_{n=1}^{\infty} E(f > r_n) \cap E(g < r_n)$$

is measurable. We can verify that

$$E(f > g) = A.$$

This gives that the set $E(f > g)$ is measarable.

(II) Let a be any real member. Then the function $g + a$ is measurable on

E. By case (I), the set $E(f > g + a) = E(f - g > a)$ is measurable. Hence the function $f - g$ is measurable.

(III) We have

$$f + g = f - (-g)$$

and $$fg = \frac{1}{4}[(f + g)^2 - (f - g)^2].$$

Since g is measurable,$-g = (-1)\,g$ is measurabe and by case (II), $f-(-g) = f + g$ is measusable on E.

Since $f + g$ and $f - g$ are measurable on E, from second identity we see that fg is measurable.

Theorem 4.5. If $\{f_n\}$ is *a* sequence of functions measurable on the set E, then each of the following four functions is measurable on E.

$$h(x) = \sup\{f_n(x) : n = 1, 2, 3, \ldots\},$$
$$g(x) = \inf\{f_n(x) : n = 1, 2, 3, \ldots\},$$

$$f^*(x) = \lim_{n\to\infty} \sup\{f_n(x)\}.$$

$$f_*(x) = \lim_{n\to\infty} \inf\{f_n(x)\}.$$

Proof : (I) Let a be any real number. We can verify that

$$E(h > a) = \bigcup_{n=1}^{\infty} E(f_n>a).$$

This gives that h is measurable on E.

(II) Let a be any real number.

We have

$$E(g < a) = \bigcup_{n=1}^{\infty} E(f_n < a).$$

So the function g is measurable on E.

(III) Let $h_k(x) = \text{Sup}\,\{f_n(x) : n = k, k + 1, k + 2, \ldots\}$

and $g_k(x) = \inf\{f_n(x) : n = k, k+1, k + 2, \ldots\}$.

Then $f^*(x) = \inf\{h_n(x) : n = 1, 2, 3, \ldots\}$,

$f_*(x) = \sup\{g_n(x) : n = 1, 2, 3, \ldots\}$,

By cases (I) and (II) the functions h_n and g_n are measurable on E ; and by the same cases the functions f^* and f_* are measurable on E.

Corollary 4.5.1. If $\{f_n\}$ is *a* sequence of functions measurable on the set E. and $f_n(x) \to f(x)$ for all $x \in E$, then the function f is measurable on E.

Since $f_n(x) \to f(x)$ at each $x \in E$,

$$\lim_{n\to\infty} \text{Sup } \{f_n(x)\} = \lim_{n\to\infty} \inf\{f_n(x)\} = f(x).$$

So $f^* = f_* = f$. Hence f is measurable on the set E.

Definition 4.2. Let E be *a* measurable set. *A* function $f : E \to \Omega$ is said to be simple if E can be expressed as

(i) $E = E_1 \cup E_2 \cup ... \cup E_n$.

where $E_1, E_2, ..., E_n$ are measurable

(ii) $E_i \cap E_j = \phi$ for $i \neq j$

(iii) $f(x) = C_i$ for $x \in E_i$ $(i = 1, 2, ..., n)$,

where $C_1, C_2, ..., C_n$ are constauts.

From difinition it follows that *a* simple function is measurable.

Theorem 4.6. Let $f : E \to \Omega^*$ be non-negative and measurable on E. Then there is an increasing sequence $\{f_n\}$ of simple functions such that

$f_n(x) \to f(x)$ for each $x \in E$.

Proof : For each positive integer n, we define the function f_n on E as follows.

$$f_n(x) = \frac{i-1}{2^n} \text{ if } \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n} \quad (i = 1, 2, 3, ..., n. 2^n)$$

$$= n \text{ if } f(x) \geq n.$$

Clearly each f_n is *a* simple function.

(I) We now show that $\{f_n\}$ is an increasing sequence.

Let $x \in E$ and n be *a* positive integer

(a) First suppose that $0 \leq f(x) < n$.

Then

$$\frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n} \quad \text{... ...} \quad (4.1)$$

for some i, $1 \leq i \leq n.\ 2^n$.

and

$$\frac{j-1}{2^{n+1}} \leq f(x) < \frac{j}{2^{n+1}} \quad \text{... ...} \quad (4.2)$$

for some j, $1 \leq j \leq (n+1).\ 2^{n+1}$.

So $\quad f_n(x) = \dfrac{i-1}{2^n}$ and $f_{n+1}(x) = \dfrac{j-1}{2^{n+1}}$.

From (4.1) and (4.2) we have

$$\frac{i-1}{2^n} < \frac{j}{2^{n+1}}$$

or $2i - 2 < j \Rightarrow 2i - 2 \le j - 1$.

$$\text{So } \frac{i-1}{2^n} \le \frac{j-1}{2^{n+1}},$$

that is, $f_n(x) \le f_{n+1}(x)$.

(b) Let $n \le f(x) < (n+1)$.

Then for some j $[1 \le j \le (n+1).\ 2^{n+1}]$,

$$\frac{j-1}{2^{n+1}} \le f(x) < \frac{j}{2^{n+1}}.$$

So $n < \dfrac{j}{2^{n+1}}$ or $n.\ 2^{n+t} < j$

or $\qquad n.\ 2^{n+1} \le (j-1)$.

We have $f_n(x)=n$ and $f_{n+1}(x) = \dfrac{j-1}{2^{n+1}}$

From above we have

$$f_n(x) \le f_{n+1}(x)$$

(c) Let $f(x) \ge n + 1$. Then

$$f_n(x) = n \text{ and } f_{n+1}(x) = (n+1).$$

$$\text{So } f_n(x) < f_{n+1}(x).$$

(II) Next, we show that $f_n(x) \to f(x)$ as $n \to \infty$ at each $x \in E$.

Let $x \in E$. Suppose that $f(x)$ is finite.

We can fined a positive integer N such that $O \le f(x) < N$.

Choose any $\varepsilon > 0$. Determine positive integer $n_0 (> N)$ with $2^{n_0} > \dfrac{1}{\varepsilon}$.

Take any $n \ge n_0$. Then $0 \le f(x) < n$. We have $\dfrac{i-1}{2^n} \le f(x) < \dfrac{i}{2^n}$ for some i, $1 \le i \le n.\ 2^n$.

$$\text{So } f_n(x) = \frac{i-1}{2^n} \text{ and}$$

$$0 \le f(x) - f_n(x) < \frac{1}{2^n} < \varepsilon.$$

Hence $f_n(x) \to f(x)$ as $n \to \infty$. If $f(x) = +\infty$, then $f_n(x) = n$ for all n. So $f_n(x) \to +\infty$ as as $n \to \infty$.

This completes the proof of the theorem.

Examples 4.1.

1. Let μ denote the Lebesgue or Lebesgue-stieltjes measure on the real line Ω and E be *a* measurable subset of Ω. If $f : E \to \Omega$ is continuous then f is measurable on E.

Solution. Let a be any real numbers. Then the set $E\,(f > a)$ is open relative to the set E. So $E\,(f > a) = E \cap G$, where G is an open set in Ω. Hence $E(f > a)$ is measurable. Since this is true for every real number a, the function f is measurable on E.

(2) Let μ be the Lebesgue measure or Lebesgue-stieltjes measure on the real line Ω. If $f : [a, b] \to \Omega$ is monotone, then f is measurable on $[a, b]$.

Solution. Since f is monotone on $[a, b]$, it is continuous on $[a, b]$ except possiblely *a* countable set. So f is measurable on $[a, b]$.

Theorem 4. 7. (Egorov's Theorem)

Let E be *a* measurable set with $\mu\,(E) < +\infty$ and $\{f_n\}$ be *a* sequence of measurable functions finite *a.e.* on E. If $\{f_n\}$ converges *a.e.* on E to *a* finite function f, then there exists, for each $\varepsilon > 0$, *a* subset A of E such that $\mu\,(E \setminus A) < \varepsilon$ and $\{f_n\}$ converges to f uniformly on A.

Proof : By removing from E, if necessary, *a* set of measure zero, we may suppose that the functions f_n are everywhere finite and $\{f_n\}$ converges to f on E.

Choose $\varepsilon > 0$ arbitrarily. We define the sets $E_1, E_2, E_3, \ldots$.by

$$E_n = \bigcap_{\nu=n}^{\infty} E(\,|f_\nu - f\,| < \varepsilon).$$

Clearly the sets $E_1, E_2, E_3, \ldots$ are measurable, $E_1 \subset E_2 \subset E_3 \subset \ldots$. and

$$E = \bigcup_{n=1}^{\infty} E_n.$$

We have

$$\lim_{n \to \infty} \mu(E_n) = \mu E\,.$$

If η be any positive number, we can find *a* positive integer N such that

$$\mu\,(E \setminus E_n) < \eta \text{ for all } n \geq N.$$

Thus for each positive integer k, there exists *a* measurable set $C_k \subset E$ and

a positive integer N_k such that $\mu(C_k) < \frac{\varepsilon}{2^k}$ and $|f_n(x) - f(x)| < \frac{1}{2^k}$ for all $x \in E \setminus C_k$ when $n \geq N_k$.

Write $C = \bigcup_{k=1}^{\infty} C_k$ and $A = E \setminus C$.

Then $E \setminus A = C = \bigcup_{k=1}^{\infty} C_k$. So,

$$\mu(E \setminus A) \leq \sum_{k=1}^{\infty} \mu(C_k) < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

Take any positive number σ. Choose positive integer k such that $\frac{1}{2^k} < \sigma$. Then

$$|f_n(x) - f(x)| < \frac{1}{2^k} < \sigma$$

for all x in A whenever $n \geq N_k$ because $A \subset E \setminus C_k$. Since N_k is independent of x, it follows that $\{f_n\}$ converges to f on A uniformly.

Convergence in measure.

Definition 4.3. Let the functions $f, f_1, f_2, f_3, ...$ be measurable and finite *a. e.* on E. If for every positive number σ

$$\lim_{n \to \infty} \mu E(f_n - f| \geq \sigma) = 0,$$

then we say that the sequence $\{f_n\}$ converges in measure to f on E. It is symbolically written as $f_n \Rightarrow f$ on E.

Theorem 4.8. Let the functions $f, f_1, f_2, f_3, ...$be measurable and finite *a.e.* on E, where $\mu(E)$ is finite. If $f_n(x) \to f(x)$ *a.e.* on E, then $f_n \Rightarrow f$ on E.

Proof : Let $A_0 = E(|f| = +\infty)$, $A_n = E(|f_n| = +\infty)$ and $B = E(f_n \not\to f)$; and $Q = \left(\bigcup_{n=0}^{\infty} A_n\right) \cup B$. Since $f, f_1, f_2 f_3, ...$ are finnite a.e. on E and $f_n(x) \to f(x)$ *a.e.* on E, we have $\mu(A_n) = 0$ $(n = 0, 1, 2, ...)$ and $\mu(B) = 0$, So $\mu(Q) = 0$.

Let σ be any positive number and let $E_k(\sigma) = E(|f_k - f| \geq \sigma)$,

$R_n(\sigma) = \bigcup_{k=n}^{\infty} E_k(\sigma)$. and $M = \bigcap_{n=1}^{\infty} R_n(\sigma)$. Clearly all the sets are measurable. Since

$R_1(\sigma) \supset R_2(\sigma) \supset R_3(\sigma) \supset \ldots$ and $\mu R_1(\sigma)$ is finite, $\mu R_n(\sigma) \to \mu(M)$. We now show that $M \subset Q$. If $x_0 \in E \setminus Q$, then

$$\lim_{k\to\infty} f_k(x_0) = f(x_0),$$

where $f(x_0), f_1(x_0), f_2(x_0), \ldots.$ are all finite. We can find *a* positive integer n such that

$$|f_k(x_0) - f(x_0)| < \sigma \text{ when } k \geq n.$$

This gives that $x_0 \notin E_k(\sigma)$ for all $k \geq n$ and so $x_0 \notin R_n(\sigma)$. This implies that $x_0 \notin M$. Thus $M \subset Q$ and so $\mu(M) = 0$. Since $E_n(\sigma) \subset R_n(\sigma)$ and $\mu R_n(\sigma) \to \mu(M) = 0$, $\mu E_n(\sigma) \to 0$ as $n \to \infty$.Since σ is arbitrary it follows that $f_n \Rightarrow f$ on E.

Theorem 4.9. If the sequence $\{f_n\}$ converges in measure to f on E, where $\mu(E) < +\infty$, then it is possible to select *a* subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k}(x) \to f(x)$ *a.e.* on E.

Proof : Let $\{\sigma_n\}$ be *a* strictly decreasing sequence of positive numbers with $\lim \sigma_n = 0$ and $\sum_{n=1}^{\infty} \eta_n$ be *a* convergent series of positive terms.

Since $\mu E(|f_n - f| \geq \sigma_1) \to 0$ as $n \to \infty$ we can find *a* positive integer n_1 such that

$$\mu E|f_{n_1} - f| \geq \sigma_1) < \eta_1.$$

Again, since $\mu E(|f_n - f| \geq \sigma_2) \to 0$ as $n \to \infty$ we can fined *a* positive integer n_2 $(>n_1)$ such that

$$\mu E(|f_{n_2} - f| \geq \sigma_2) < \eta_2.$$

Continuing the above process we obtain *a* sequence $\{n_k\}$ $(n_1 < n_2 < n_3 < \cdots)$ of positive integers such that

$$\mu E(|f_{n_k} - f| \geq \sigma_k) < \eta_k \ (k = 1, 2, 3, \ldots).$$

Let $R_k = \bigcup_{i=k}^{\infty} E(|f_{n_i} - f| \geq \sigma_i)$ and $Q = \bigcap_{k=1}^{\infty} R_k$.

Since $R_1 \supset R_2 \supset R_3 \supset \ldots.$ and μR_1 is fimite,

$$\lim_{k\to\infty} \mu R_k = \mu Q.$$

Now

$$\mu R_k \leq \sum_{i=k}^{\infty} \mu E(|f_{n_i} - f| \geq \sigma_i) < \sum_{i=k}^{\infty} \eta_i.$$

Since $\sum_{i=1}^{\infty} \eta_i$ is convergent, $\sum_{i=k}^{\infty} \eta_i \to 0$ as $k \to \infty$. Hence $\mu R_k \to 0$ as $k \to \infty$ and so $\mu Q = 0$.

We show that $f_{n_k}(x) \to f(x)$ for $x \in E/Q$.

Let $x \in E/Q$. Then $x \notin Q$ which gives that $x \notin R_{k_0}$ for some positive integer k_0. So

$x \notin E\ (|f_{n_k} - f| \geq \sigma_k)$ for all $k \geq k_0$. and hence for all $k \geq k_0$

$$|f_{n_k}(x) - f(x)| < \sigma_k.$$

Since $\sigma_k \to 0$ as $k \to \infty$, we have

$$\lim_{k \to \infty} f_{n_k}(x) = f(x).$$

This completes the proof of the theorem.

Theorem 4.10. Let the functions $f_1, f_2, f_3, \ldots$ be measurable and finite *ae.* on E, where $\mu E < +\infty$. If for every positive number σ,

$$\lim_{m,n \to \infty} \mu E\ (|f_m - f_n| \geq \sigma) = 0,$$

then there is a function f measurable and finite *a. e.* on E such that $\{f_n\}$ converges in measure to f on E.

Proof : By removing from E, if necessary, *a* set of measure zero, we may suppose that $f_1, f_2, f_3, \ldots$ are all finite on E.

Let k be any positive integer. Then since

$$\lim_{m,n \to \infty} \mu E\ (|f_m - f_n| \geq \frac{1}{2^k}) = 0,$$

we can find *a* positive integer n_k such that

$$\mu E(|f_m - f_n| \geq \frac{1}{2^k}) < \frac{1}{2^k} \text{ when } m, n > n_k. \quad \ldots \quad \ldots \quad (4.3)$$

We may choose the sequence $\{n_k\}$ such that $n_1 < n_2 < n_3 < \ldots$

Write $g_k = f_{n_k}$ $(k = 1, 2, 3, \ldots)$.

Then from (4.3) we have

$$\mu E(\,|\,g_{k+1}-g_k|\geq \frac{1}{2^k})<\frac{1}{2^k}\ (k=1,2,3,\,....)\ \ldots\quad \ldots \tag{4.4}$$

Let $A_k = \bigcup_{i=k}^{\infty} E(\,|\,g_{k+1} - g_k\,| \geq \frac{1}{2^k})$

and $A = \bigcap_{k=1}^{\infty} A_k$.

Then $A_1 \supset A_2 \supset A_3 \supset \ldots$ and $\mu\, A \leq \mu\, E < +\infty$.

So,

$$\mu\, A = \lim_{k\to\infty} \mu\, A_k. \quad \ldots \quad \ldots \tag{4.5}$$

We have

$$\mu A_k \leq \sum_{i=k}^{\infty} \mu\, E(\,|\,g_{i+1} - g_i\,| \geq \frac{1}{2^i})$$

$$< \sum_{i=k}^{\infty} \frac{1}{2^i} = \frac{1}{2^{k-1}}\ \text{[By (4.4)]}$$

This gives that

$$\lim_{k\to\infty} \mu\, A_k = 0$$

and from (4.5) we get $\mu A = 0$.

Let $x \in E \setminus A$. Then $x \notin A_{k_0}$ for some positive intger k_0. Since

$A_{k_0} = \bigcup_{k=k0}^{\infty} E\,(\,|g_{k+1} - g_k\,| \geq \frac{1}{2^k})$, it follows that

$x \notin E(\,|\,g_{k+1} - g_k\,| \geq \frac{1}{2^k})$ for all $k \geq k_0$. that is,

$$|\,g_{k+1}(x) - g_k(x)\,| < \frac{1}{2^k} \text{ for all } k \geq k_0. \quad \ldots \quad \ldots \tag{4.6}$$

Take any two positive integers γ, s with $\gamma > s \geq k_0$. Then

$$|g_\gamma(x) - g_s(x)| \leq \sum_{k=s}^{\gamma} |g_{k+1}(x) - g_x(x)|$$

$$< \sum_{k=s}^{\gamma} \frac{1}{2^k} < \frac{1}{2^{s-1}} \quad \dots \quad \dots \quad (4.7)$$

This shows that $\{g_k(x)\}$ is convergent.

Let

$$g(x) = \lim_{k \to \infty} g_k(x).$$

Letting $\gamma \to \infty$ in (4.7) we get

$$|g(x) - g_s(x)| \leq \frac{1}{2^{s-1}} = \frac{2}{2^s}.$$

Take any $n \geq n_{k_0}$ and $k \geq k_0$. From (4.3) we have

$$|f_n(x) - f_{n_k}(x)| < \frac{1}{2^k}.$$

$$|f_n(x) - g(x)| \leq |f_n(x) - f_{n_k}| + |g_k(x) - g^{(x)}|$$

$$< \frac{1}{2^k} + \frac{2}{2^k} = \frac{3}{2^k}.$$

So $f_n(x) \to g(x)$ as $n \to \infty$. Thus $\{f_n\}$ converges *a. e.* on *E*. Therefore, by Theorem 4·8, $\{f_n\}$ converges in measure to g on *E*.

Theorem 4.11. Let μ be *a* stieltjes measure on the real line Ω and *E be* a measurable (μ) subset of Ω with $\mu E < +\infty$. If the function f is measurable and finite *a. e.* on *E*, then given any pair of positive numbers σ, ε there is *a* function g continuous on Ω such that

$$\mu E(|f - g| \geq \sigma) < \varepsilon.$$

Proof : (I) We first suppose that f is bounded on the set *E*. Let *A* and *B* denote respectively the glb and lub of f on *E*. Take any subdivision

$$A = y_0 < y_1 < y_2 < \dots < y_n = B$$

of $[A, B]$ with $\max(y_i - y_{i-1}) < \sigma$.

Write

$$E_i = E(y_{i-1} \leq f < y_i) \ (i = 1,2, \ldots, n-1)$$

and $E_n = E(y_{n-1} \leq f \leq y_n)$.

Then the sets $E_1, E_2, \ldots, E_n$ are measurable and $E = E_1 \cup E_2 \cup \ldots \cup E_n$; $E_i \cap E_j = \phi$ for $i \neq j$.

For each i we choose *a* closed set $F_i \subset E_i$ with $\mu(E_i \setminus F_i) < \frac{\varepsilon}{n}$.

Write $F = F_1 \cup F_2 \cup \ldots \cup F_n$.

Then F is bounded, closed and $E \setminus F \subset \bigcup_{i=1}^{n} (E_i \setminus F_i)$.

Let $[c, d]$ be the smallest closed interval containing the set F. Then $c,d \in F$. *Write* $G = [c, d] \setminus F = (c, d) \setminus F$ Then G is an open set. We can write G as

$G = \cup_i (a_i, b_i)$, where the intervals (a_i, b_i) are pairwise disjoint.

We now define the function g on Ω as follows.

$g(x) = y_i$ if $x \in F_i$ $(i = 1, 2, \ldots, n)$

$= g(a_i) + \dfrac{x-a_i}{b_i-a_i} [g(bi) - g(a_i)]$ if $x \in (a_i, b_i)$ $(i = 1,2, \ldots)$

$= g(c)$ if $x < c$

$= g(d)$ if $x > d$.

Clearly g is continuous at each point of the set $G \cup (-\infty, c) \cup (d, \infty)$. If $\alpha \in F$, then $\alpha \in F_i$ for some i. Since the sets $F_1, F_2, \ldots F_n$ are pairwise disjoint there is an $\eta > 0$ such that $F \cap (\alpha-\eta, \alpha+\eta) = F_i \cap (\alpha-\eta, \alpha+\eta)$. If α is an isolated point of F on the right, then $\alpha = a_j$ for some j. This gives that g is continuous on the right at α. If α is *a* limit point of F on the right, then either $(\alpha, \alpha+\delta) \subset F_i$ for some δ $(0 < \delta \leq \eta)$ or α is *a* limit point of the set $\{a_1, b_1, a_2, b_2, \ldots\}$ on the right. In both cases $g(x) = y_i$ for all x in $(\alpha, \alpha+\delta)$ for some δ $(0 < \delta \leq \eta)$. Hence g is continuous at α on the right. Similarly we can show that g is continnous at α on the left. So g is continuous at α. Further $A \leq g(x) \leq B$ for all $x \in \Omega$.

If $x \in F$, Then $x \in F_i$ for some i.

So $y_{i-1} \leq f(x) \leq y_i$ which gives that $|f(x) - g(x)| < \sigma$. Therefore,

$$E(|f-g| \geq \sigma) \subset E \setminus F.$$

So $\mu E(|f-g| \geq \sigma) < \varepsilon$.

(II) Suppose that f is not bounded on E. Write $E_0 = E\,(|f| = +\infty)$, $E_k = E\,(0 \leq |f| < k)$. $(k = 1, 2, 3, \ldots)$. Then the sets $E_0, E_1, E_2, E_3, \ldots$ are measurable $E_1 \subset E_2 \subset E_3 \subset \ldots$ and $\bigcup_{k=1}^{\infty} E_k = E \setminus E_0$; and $\mu(E_0) = 0$. We can choose *a* positive integer k such that

$$\mu(E \setminus E_k) < \frac{1}{2}\varepsilon.$$

By case (I), we can determine *a* function g continuous on Ω such that

$$\mu\, E_k\,(|f - g| \geq \sigma) < \frac{1}{2}\varepsilon.$$

Now

$$E(|f - g| \geq \sigma) \subset E_k\,(|f - g| \geq \sigma) \cup (E \setminus E_k).$$

So $\mu\, E(|f - g| \geq \sigma) \leq \mu\, E_k\,(|f - g| \geq \sigma) + \mu\,(E / E_k) < \varepsilon$.

This completes the proof of the theorem.

Theorem 4.12. Let μ be the Lebesgue or Lebesgue-Stieltjes measure on the real line Ω and E be *a* measurable (μ) subset of Ω with $\mu(E) < +\infty$. If f *is* measurable and finite *a. e.* on E, then there exists *a* sequence $\{f_n\}$ of functions continuous on Ω such that $f_n(x) \to f(x)$ *a. e.* on E.

Proof : Let $\{\sigma_n\}$ and $\{\varepsilon_n\}$ be two strictly decreasing sequences of positive numbers with $\lim \sigma_n = 0$ and time $\varepsilon_n = 0$.

For each pair σ_n, ε_n there is *a* function g_n continuous on Ω such that

$$\mu\, E(|f - g_n| \geq \sigma_n) < \varepsilon_n.$$

Choose any positive number σ. There is *a* positive integer n_0 such that

$$\sigma_n < \sigma \text{ when } n \geq n_0.$$

Take any $n \geq n_0$. Then

$$|f(x) - g_n(x)| \geq \sigma \text{ implies } |f(x) - g_n(x)| > \sigma_n.$$

which gives that

$$E(|f - g_n| \geq \sigma) \subset E\,(|f - g_n| \geq \sigma_n).$$

So. $$\mu\, E\,(|f - g_n| \geq \sigma) < \varepsilon_n \text{ for all } n \geq n_0.$$

$$\therefore \quad \lim_{n \to \infty} \mu\, E(|f - g_n| \geq \sigma) = 0.$$

Hence the sequence $\{g_n\}$ converges in measure to f on E. So by Theorem 4.8, There is *a* subsequence $\{g_{n_k}\}$ such that

$$g_{n_k}(x) \to f(x) \text{ a.e. on } E.$$

This proves the theorem.

Theorem 4.13. Let μ denote the Lebesgue or Lebesgue-Stieltjes measure

on the real line Ω and let $f: E \to \Omega^*$ be measurable and finite *a.e.* on $E \subset \Omega$ with $\mu E < +\infty$. Then for $\varepsilon > 0$ there is *a* measurable set $A \subset E$ with $\mu(E \setminus A) < \varepsilon$ such that f is continuous on A relative to the set A.

Proof : By Theorem 4.12 there is *a* sequence $\{f_n\}$ of functions continuous on Ω such that $f_n(x) \to f(x)$ *a.e.* on E. Choose any $\varepsilon > 0$. By Egoroff's Theorem there is *a* measurable subset A of E with $\mu(E \setminus A) < \varepsilon$ such that $\{f_n\}$ converges uniformly to f on the set A.

We now show that f is continuous on A relative to the set A.

Let η be any positive number. Then there is *a* positive integer N such that for all x in A,

$$| f_n(x) - f(x) | < \frac{1}{3}\eta \text{ when } n \geq N.$$

Take any point α in A. The function f_N is continuous at α. So there is *a* $\delta > 0$ such that

$$| f_N(x) - f_N(\alpha)| < \frac{1}{3}\eta \text{ for all } x \text{ in } (\alpha - \delta, \alpha + \delta).$$

Let $x \in A \cap (\alpha - \delta, \alpha + \delta)$. We have

$$| f(x) - f(\alpha)| \leq | f(x) - f_N(x)| + |f_N(x) - f_N(\alpha)| + | f_N(\alpha) - f(\alpha) |$$
$$< \frac{1}{3}\eta + \frac{1}{3}\eta + \frac{1}{3}\eta = \eta.$$

Hence f is continuous at α relative to the set A. Since α is arbitrary, f is continuoues on A relative to A.

CHAPTER—V

LEBESGUE INTEGRAL OF BOUNDED FUNCTIONS

Let $(X, \mathcal{S}, \mu)$ be a measure space and let $E \in \mathcal{S}$ with $\mu E < +\infty$. Suppose that $f: E \to \Omega$ be a bounded function, where Ω denotes the set of all real numbers. A family $P = \{E_1, E_2, ..., E_n\}$ of subsets of E is said to be a measurable partition of E if following conditions hold.

(i) $E_i \in \mathcal{S}$ $(i = 1, 2, .., n)$ (ii) $E_i \cap E_j = \phi$ for $i \neq j$ (iii) $E = E_1 \cup ... \cup E_n$.

A measurable partition $Q = \{F_1, F_2, ..., F_m\}$ of E is said to be a refinement of P if each $F_i \subset E_j$ for some j and we write $Q \geq P$ or $P \leq Q$. From definition it follows that a refinement Q of P is obtained from P by partitioning each component of P into several components. We denote by Δ_E the collection of all measurable partitions of E. We can verify the following.

(a) $P \geq P$ for every $P \in \Delta_E$.

(b) If P, Q, R are in Δ_E, then $P \leq Q$ and $Q \leq R$ imply that $P \leq R$.

(c) If P, Q are in Δ_E, then there is an element R in Δ_E with $R \geq P$ and $R \geq Q$.

From above we see that $(\Delta_E, \geq)$ is a directed set.

Take any measurable partition $P = \{E_1, E_2, ..., E_n\}$ of E. Denote by A_i and B_i the glb and lub respectively of f on E_i. Now form the sums

$$s_P(f) = \sum_{i=1}^{n} A_i\, \mu(E_i) \text{ and } S_P(f) = \sum_{i=1}^{n} B_i\, \mu(E_i).$$

$s_P(f)$ and $S_P(f)$ are called lower and upper Lebesgue sums of f for the partition P.

We have

$$A\, \mu(E) \leq s_P(f) \leq S_P(f) \leq B\, \mu(E) \qquad \ldots \qquad \ldots \qquad (5.1),$$

where A and B denote respectively the glb and lub of f on E. Consider the nets

$\{s_P(f) : P \in \Delta_E\}$ and $\{S_P(f) : P \in \Delta_E\}$.

From (5.1) we see that they are bounded.

Let
$$L = \text{lub } \{s_P(f) : P \in \Delta_E\},$$
$$U = \inf \{S_P(f) : P \in \Delta_E\}.$$

The numbers L and U are called the lower and upper Lebesgue integrals respectively of f on E. If $L = U = I$ (say), then we say that f is integrable in Lebesgue sense or integrable (L) or simply integrable on E and write

$$I = (L) \int_E f d\mu \text{ or } I = \int_E f d\mu.$$

Lemma 5.1. Let P and Q be two measurable partitions of E with $P \le Q$. Then

$s_P(f) \le s_Q(f)$ and $S_P(f) \ge S_Q(f)$.

Proof : Let $P = \{E_1, E_2, \dots, E_n\}$. Since $P \le Q$, it follows that Q is obtained from P by partitioning one or more components of P into several components. For simplicity we suppose that Q is obtained from P by partitioning only E_1 into p components $E_1^1, E_1^2, \dots, E_1^P$. Thus $Q = \{E_1', E_1^2, \dots, E_1^P, E_2, E_3, \dots E_n\}$, where

$E_1 = E_1^1 \cup E_1^2 \cup \dots \cup E_1^P;\ E_1^r \cap E_1^s = \phi$ for $r \neq s$.

Let $A_1^k = \inf\{f(x) : x \in E_1^k\}$,

$B_1^k = \sup\ \{f(x) : x \in E_1^k\}$.

Since $E_1^k \subset E_1$, $A_1^k \ge A_1$ and $B_1^k \le B_1$,

We have

$$s_Q(f) = \sum_{k=1}^{p} A_1^k \mu\left(E_1^k\right) + \sum_{i=2}^{n} A_i \mu(E_i)$$

$$\ge \sum_{k=1}^{p} A_1 \mu\left(E_1^k\right) + \sum_{i=2}^{n} A_i \mu(E_i)$$

$$= A_1\ \mu(E_1) + \sum_{k=2}^{p} A_i\ \mu\left(E_i\right) = s_P(f).$$

$$S_Q(f) = \sum_{k=1}^{p} B_1^k \mu\left(E_i^k\right) + \sum_{i=2}^{n} B_i \mu(E_i)$$

$$\le \sum_{k=1}^{p} B_1 \mu\left(E_1^k\right) + \sum_{i=2}^{n} B_i\ \mu(E_i)$$

$$= B_1\ \mu(B_1) + \sum_{i=2}^{n} B_i\ \mu\left(E_i\right) = S_P(f).$$

This proves the Lemma.

From Lemma 5.1, we see that the net $\{s_p(f) : P \in \Delta_E\}$ is increasing and the net $\{S_P(f) : P \in \Delta\ E\}$ is decreasing ; also they are bounded. Hence each of them is convergent. We have

$$\lim_{P} \{s_P(f) : P \in \Delta_E\} = \text{lub}\ \{s_P(f) : P \in \Delta_E\} = L$$

and $\lim\limits_{P} \{s_P(f) : P \in \Delta_E\} = \inf \{s_P(f) : P \in \Delta_E\} = U.$

Lemma 5.2. $L \leq U.$

Since $s_P(f) \leq S_P(f)$ for all $P \in \Delta_E$, taking limit we obtain

$$L \leq U.$$

Lemma 5.3. Let P and Q be any two elements in Δ_E. Then

$$s_P(f) \leq S_Q(f) \text{ and } s_Q(f) \leq S_P(f).$$

Since $s_P(f) \leq L \leq U \leq S_Q(f)$ and

$$s_Q(f) \leq L \leq U \leq S_P(f)$$

the result follows.

Theorem 5.1. Let $f : E \to \Omega$ be bounded on the measurable set E with $\mu E < +\infty$. Then *a* necessary and sufficiunt condition for integrability of f on E is that for any $\varepsilon > 0$ there exists *a* measurable partition P of E such that

$$S_P(f) - s_P(f) < \varepsilon.$$

Proof. First we suppose that f is integrable on E. Then

$$L = U = \int_E f d\mu = I \text{ (say).}$$

Choose any $\varepsilon > 0$. There are measurable partitions P_1 and P_2 of E with

$$s_{P_1}(f) > I - \frac{1}{2}\varepsilon \text{ and } S_{P_2}(f) < I + \frac{1}{2}\varepsilon \qquad \ldots \qquad (5.2)$$

Now Choose *a* measurable partition P of E with $P \geq P_1$ and $P \geq P_2$. Then

$$s_{P_1}(f) \leq s_P(f) \leq S_P(f) \leq S_{P_2}(f) \qquad \ldots \qquad (5.3)$$

From (5.2) and (5.3) we get

$$I - \frac{1}{2}\varepsilon < s_P(f) \leq S_p(f) < I + \frac{1}{2}\varepsilon.$$

$$\text{So } S_P(f) - s_P(f) < \varepsilon. \qquad \ldots \qquad \ldots \qquad (5.4)$$

Next, suppose that the given condition holds. Choose any $\varepsilon > 0$. There is *a* measurable partition P of E satisfying the condition (5.4). Now

$$s_P(f) \leq L \leq U \leq S_P(f) . \qquad \ldots \qquad \ldots \qquad (5.5)$$

From (5.4) and (5.5) we get

$$0 \leq U - L < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary it follows that $L = U$. So the function f is integrable on E.

Corollary 5.1. Let the function f be integrable on the set E and F be *a* measurable subset of E. Then f is integrable on F.

Theorem 5.2. Let E be *a* measurable set with $\mu E < +\infty$. If $f: E \to \Omega$ is bounded and measurable on E, then f is integrable on E.

Proof : Let A and B be the glb and lub respectively of f on E. If $A = B$, then $f(x) = A$ for all $x \in E$. So f is integrable on E. Suppose that $A < B$.

Choose any $\varepsilon > 0$. Take any subdivision $A = y_0 < y_1 < y_2 < \ldots y_n = B$ of the interval $[A, B]$ with

$$\max (y_i - y_{i-1}) < \varepsilon/(1+ \mu E).$$

Let $E_i = E\,(y_{i-1} \le f < y_i)$ $(i = 1,2, \ldots, n-1)$

$$E_n = E(y_{n-1} \le f \le y_n).$$

Since f is measurable on E, the sets $E_1, E_2, \ldots, E_n$ are measurable and $E_i \cap E_j = \phi$ for $i \neq j$. So $P = \{E_1, E_2, \ldots, E_n\}$ is *a* measurable partition of E.

Let $\quad A_i = \text{glb}\ \{f(x) : x \in E_i\},$

$\quad\quad B_i = \text{lub}\ \{f(x) : x \in E_i\}$

Then $\quad y_{i-1} \le A_i \le B_i \le y_i$ and so

$\quad\quad B_i - A_i \le y_i - y_{i-1} < \varepsilon / (1+\mu E).$

We have

$$S_P(f) - s_P(f) = \sum_{i=1}^{n} (B_i - A_i)\mu E_i$$

$$< \frac{\varepsilon}{1+\mu E} \cdot \sum_{i=1}^{n} \mu E_i = \frac{\mu E}{1+\mu E} \cdot \varepsilon < \varepsilon.$$

Hence f is integrable on E.

Theorem 5.3. Let $f: [a, b] \to \Omega$ be bounded. If f integrable on $[a, b]$ in Riemann sense, then it is intgrable on $[a, b]$ in Lebesgue sense also and the two integrals are equal.

[Here the measure μ is the usual Lebesgue measure on Ω}.

Proof : Suppose that the function f is integrable on $[a, b]$ in Riemann sense and let

$$I = (R) \int_a^b f(x)dx.$$

Let $D : (a = x_0 < x_1 < x_2 < \ldots < x_n = b)$ be any subdivision of $[a, b]$.

Write $\quad E = [a, b],$

$\quad\quad E_i = [x_{i-1}, x_i)$ $(i = 1,2, \ldots n - 1)$

and $\quad E_n = [x_{n-1}, x_n].$

Then $P = \{E_1, E_2, \ldots, E_n\}$ is *a* meaurable partition of E.

Let $m_i = glb\{f(x) ; x_{i-1} \leq x \leq x_i\}$,

$M_i = lub \{f(x) : x_{i-1} \leq x \leq x_i\}$

and $A_i = glb\{f(x) : x \in E_i\}$,

Then $m_i \leq A_i \leq B_i \leq M_i$

So $$\sum_{i=1}^{n} m_i(x_i - x_{i-1}) \leq s_P(f) \leq S_P(f) \leq \sum_{i=1}^{n} M_i (x_i - x_{i-1}) \quad \text{... (5.6)}$$

Choose any $\varepsilon > 0$. Since f is R-integrable on $[a, b]$ we can choose the subdivision D such that

$$\sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) < \varepsilon \quad \text{... ... (5.7)}$$

From (5.6) and (5.7) we have

$$0 \leq U - L \leq S_P(f) - s_P(f) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain $L = U$. So f is integrable on E in Lebesgue sense.

Letting max $(x_i - x_{i-1}) \to 0$ in(5.6) we get

$$L = U = I.$$

Hence $(L) \int_E f d\mu = (R) \int_a^b f(x) dx.$

Definition of Lebesgue Integral as the limit of a sum.

Let E be a measurable (μ) set with $\mu E < +\infty$ and let $f : E \to \Omega$ be bounded. Let $P = \{E_1, E_2, ..., E_n\}$ be a measurable partition of E. We now form the sum

$$\sigma_p(f) = \sum_{1=1}^{n} f(\xi_i)\mu(E_i)$$

where $\xi_i \in E_i$ $(i = 1, 2, ..., n)$.

Denote by A and B the glb and lub of f on E. Since $A \leq f(x) \leq B$ for $x \in E$, we have

$$A \sum_{1=1}^{n} \mu(E_i) \leq \sum_{i=1}^{n} f(\xi_i)\mu(E_i) \leq B \sum_{i=1}^{n} \mu(E_i)$$

$$\text{or } A\mu(E) \leq \sigma_P(f) \leq B\,\mu(E) \quad \text{... .. (5.8)}$$

From (5.7) we see that the net

$$\{\sigma_P(f) : P \in \Delta_E\} \quad \text{... ... (5.9)}$$

is bounded. If the net (5.8) converges to a number J(say) irrespective of the choice of the points $\xi_1, \xi_2, \ldots, \xi_n$ we say that f is integrable in Lebesgue sense or integrable (L) or simply integrable on E and we write

$$J = \lim_{P} \{\sigma_P(f) : P \in \Delta_E\} = (L) \int_E f d\mu;$$

and J is called the integral of f on E.

We have given two definitions of Lebesgue integral on the set E.

In the next theorem we will show that the two definitions are equivalent.

Theorem 5.4. Two definitions of Lebesgue integrals of a bounded function on a measurable (μ) set E with $\mu E < +\infty$ are equivalent.

Proof: Let E be a measurable set with $\mu E < +\infty$ and $f : E \to \Omega$ be bounded. Suppose that f is integrable on E according to the first defintion.

Choose any $\varepsilon > 0$. Write $I = (L) \int_E fd\mu$. Then there is a measusable partition P_0 of E such that for all measurable partitions P of E with $P \geq P_0$,

$$I - \frac{1}{2}\varepsilon < s_P(f) \leq S_P(f) < I + \frac{1}{2}\varepsilon \qquad \ldots \qquad \ldots \qquad (5.10)$$

Choose any points $\xi_i \in E_i$ $(i = 1, 2, \ldots, n)$. Then clcarly

$$s_P(f) \leq \sum_{i=1}^{n} f(\xi_i)\, \mu(E_i) \leq S_P(f) \qquad \ldots \qquad \ldots \qquad (5.11)$$

From (5.9) and (5.10) we get

$$|\sigma_P(f) - I| < \varepsilon \text{ for all } P \text{ in } \Delta_E \text{ with } P \geq P_0.$$

This gives that the net $\{\sigma_P(f) : P \in \Delta_E\}$converges to the number I. So the function f is integrable on E according to the second definition.

Next, suppose that f is integrable on E according to the second definition.

Write $J = (L) \int_E fd\mu$.

Choose any $\varepsilon > 0$. Then there is a measurable partition P_0 of E such that for all P in Δ_E with $P \geq P_0$,

$$|\sigma_P(f) - J| < \frac{1}{4}\varepsilon.$$

or

$$J - \frac{1}{4}\varepsilon < \sigma_P(f) < J + \frac{1}{4}\varepsilon \qquad \ldots \qquad \ldots \qquad (5.12)$$

Take any measurable partition $P = \{E_1, E_2, \ldots, E_n\}$ of E with $P \geq P_0$. Denote by A_i and B_i the glb and lub respectively of f on E_i. Then we can find points ξ_i and η_i in E_i such that

$$\left.\begin{aligned} f(\xi_i) &< A_i + \frac{\varepsilon}{4(1+\mu E)} \\ f(\eta_i) &> B_i - \frac{\varepsilon}{4(1+\mu E)} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (5.13)$$

Write

$$\sigma'_P(f) = \sum_{i=1}^{n} f(\xi_i)\mu E_i$$

and $$\sigma_P''(f) = \sum_{i=1}^{n} f(\eta_i)\mu E_i \;;$$

Using (5.11) and (5.12) we get,

$$\sigma_P'(f) < s_P(f) + \frac{\varepsilon\mu E}{4(+\mu E)} < s_P(f) + \frac{\varepsilon}{4},$$

$$\sigma_P''(f) > S_P(f) - \frac{\varepsilon\mu E}{4(1+\mu E)} > S_P(f) - \frac{\varepsilon}{4},$$

So $$s_P(f) > J - \frac{1}{2}\varepsilon.$$

and $$S_P(f) < J + \frac{1}{2}\varepsilon.$$

Hence $$J - \frac{1}{2}\varepsilon < s_P(f) \leq S_p(f) + \frac{1}{2}\varepsilon \quad \ldots \quad (5.14)$$

and so $S_P(f) - s(f) < \varepsilon$.

Therefore the function f is integrable on E according to the first definition. From (5.14) we have

$$J - \frac{1}{2}\varepsilon < I < J + \frac{1}{2}\varepsilon.$$

Since $\varepsilon > 0$ arbitrary we obtain

$$I = J.$$

Theorem 5.5. Let E be *a* measurable set with $\mu E < +\infty$ and $f, g : E \to \Omega$ be bounded and intgrable on E. Then the following hold.

(1) $A\mu E \leq \int_E f d\mu \leq B \mu E$.

(2) If $\mu E = 0$, then $\int_E fd\,\mu = 0$.

(3) If $A = B$, then

$$\int_E fd\,\mu = A.\,\mu E.$$

(4) If $f(x) \geq 0$ on E, then $\int_E fd\,\mu \geq 0$.

(5) $\lim\limits_P s_P(f) = \int_E fd\,\mu = \lim\limits_P S_P(f)$.

(6) If k is any real number, then

$$\int_E (kf)d\mu = k\int_E fd\,\mu.$$

(7) $f \pm g$ are integrable on E and

$$\int_E (f \pm g)\,d\mu = \int_E fd\,\mu \pm \int_E fd\,\mu.$$

Above results follow immediately from definition.

Theorem 5.6. Let the function f and g be bounded and integrable on the set E. Then fg is integrable on E.

Proof : Since f and g are bounded on E there is *a* positive member M such that

$$|f(x)| \leq M \text{ and } |g(x)| \leq M \text{ for all } x \in E.$$

Choose any $\varepsilon > 0$. Since f and g are integrable on E there is *a* measurable partition $P = \{E_1, E_2, \ldots, E_n\}$ of E such that

$$S_P(f) - s_P(f) < \frac{\varepsilon}{2M} \text{ and } S_P(g) - s_P(g) < \frac{\varepsilon}{2M}.$$

Let A_i', A_i'', A_i denote respectively the glb of f, g, fg on E_i and B_i', B_i'', B_i denote the lub of f, g, fg on E_i.

For any two points x', x'' in E, we have

$$\begin{aligned}
&|f(x')g(x') - f(x'')\,g\,(x'')| \\
&\leq |g(x')|\,|f(x') - f(x'')| + |f(x'')|\,|g(x') - g(x'')| \\
&\leq M\,\{B_i' - A_i') + (B_i''-A_i'')\}.
\end{aligned}$$

This implies that

$$B_i - A_i \leq M\,\{(B_i'-A_i') + (B_i''-A_i'')\}.$$

So

$$\begin{aligned}
S_P(fg) - s_P(fg) &\leq M\,[\{S_P(f) - s_P(f)\}+\{S_P(g) - s_P(g)\}] \\
&< M\left(\frac{\varepsilon}{2M}+\frac{\varepsilon}{2M}\right) = \varepsilon.
\end{aligned}$$

Hence fg is integrable on E.

Theorem 5.7. Let the function f be bounded and integrable on the set E. Then $|f|$ is integrable on E and

$$\left|\int_E f d\mu\right| \le \int_E |f| d\mu.$$

Proof : Choose any $\varepsilon > 0$. Since f is integrable on E there is *a* measurable partition $P = \{E_1, E_2, \ldots, E_n\}$ of E such that

$$S_P(f) - s_P(f) < \varepsilon.$$

Let A_i A_i' denote the glb of f and $|f|$ respectively on E_i and B_i, B_i' denote the lub of f and $|f|$ on E_i. For any x', x'' in E_i,

$$\big||f(x')| - |f(x'')|\big| \le |f(x') - f(x'')|,$$

which gives that

$$B_i' - A_i' \le B_i - A_i.$$

So $\qquad S_P(|f|) - s_P(|f|) \le S_P(f) < \varepsilon.$

Hence $|f|$ is integrable on E.

Again, let $P = \{E_1, F_2, \ldots, E_n\}$ be any measurable partition of E.

Since $|f(x)| \le B_i'$ for all $x \in E_i$, we have

$$|B_i| \le B_i'.$$

So
$$|S_P(f)| = \left|\sum_{i=1}^{n} B_i \mu(E_i)\right|$$

$$\le \sum_{i=1}^{n} |B_i| \mu(E_i)$$

$$\le \sum_{i=1}^{n} B_i' \mu(E_i) = S_P(|f|)$$

Now taking limit we obtain,

$$\left|\int_E f d\mu\right| \le \int_E |f| d\mu.$$

Theorem 5.8. Let the function f be bounded and integrable on the set E with $\mu E < +\infty$. If the set E can be expressed as the union of *a* countable family of pairwise disjoint sets $E_1, E_2, E_3, \ldots\ldots$, then

$$\int_E f d\mu = \sum_i \int_{E_i} f d\mu.$$

Proof. (I) First we consider the case when $E = E_1 \cup E_2$, where E_1, E_2 are measurable and $E_1 \cap E_2 = \phi$.

write $\quad I_1 = \int_{E_1} fd\,\mu, I_2 = \int_{E_2} fd\mu$ and $I = \int_E fd\,\mu$.

Choose any $\varepsilon > 0$. There is *a* measurable partition P_1 of E_1 and *a* measurable partition P_2 of E_2 such that

$$I_1 - \frac{1}{2}\varepsilon < s_{P_1}(f : E_1) \le S_{P_1}(f : E_1) < I_1 + \frac{1}{2}\varepsilon$$

$$I_2 - \frac{1}{2}\varepsilon < s_{P_2}(f : E_2) \le S_{P_2}(f : E_2) < I_2 + \frac{1}{2}\varepsilon.$$

Let $P = P_1 \cup P_2$. Then P is *a* measurable partition of E. We have

$$s_P(f : E) = s_{P_1}(f : E_1) + s_{P_2}(f : E_2) > I_1 + I_2 - \varepsilon,$$

$$S_P(f : E) = S_{P_1}(f : E_1) + S_{P_2}(f : E_2) < I_1 + I_2 + \varepsilon.$$

Since $s_P(f : E) \le I \le S_P(f : E)$ we have

$$I_1 + I_2 - \varepsilon < I < I_1 + I_2 + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$I = I_1 + I_2.$$

(II) Let $E = E_1 \cup E_2 \cup ... \cup E_n$, where $E_1, E_2, E_3, ..., E_n$ are measurable and pairwise disjoint. Using step (*I*) we can show that

$$\int_E fd\,\mu = \sum_{i=1}^{n} \int_{E_i} fd\mu.$$

(III) Lastly, let $E = \bigcup_{i=1}^{\infty} E_i$, where $E_1, E_2, E_3, ..., E_n, ...$ are measurable and pairwise disjoint. Write

$$E^{(n)} = E_1 \cup E_2 \cup \cdots \cup E_n \text{ and } R_n = \bigcup_{i=n+1}^{\infty} E_i .$$

Then $E = E^{(n)} \cup R_n$ and E^n, R_n are measurable and disjoint. So by above

$$\int_E fd\mu = \int E^{(n)} fd\mu + \int_{R_n} fd\mu$$

$$= \sum_{i=1}^{n} \int_{E_i} fd\mu + \int_{R_n} fd\mu. \qquad ... (5.15)$$

Denote by A and B the glb and lub of f on E. Then $A \le f(x) \le B$ for $x \in R_n$ which gives that

$$A \mu R_n \le \int_{R_n} fd\mu \le B \mu R_n \qquad ... \quad ... \quad (5.16)$$

Since $\quad \mu E = \sum_{i=1}^{\infty} \mu E_i, \mu R_n = \sum_{i=n+1}^{\infty} \mu E_i$

and μE is finite, it follows that $\mu R_n \to 0$ as $n \to \infty$. Hence from (5.16) we see that

$$\lim_{n\to\infty} \int_{R_n} f d\mu = 0. \qquad \ldots \qquad \ldots \qquad (5.17)$$

Combining (5.15) and (5.17) we obtain

$$\int_E f d\mu = \sum_{i=1}^{\infty} \int_{E_i} f d\mu.$$

Corollary 5.8.1. Let the function f be non-negative, bounded and measurable on the set E with $\mu E < +\infty$. If $\int_E f d\mu = 0$, then $f(x) = 0$ *a.e.* on E.

Proof : Denote by B the lub of f on E. We may suppose that $B > 0$. If $B = 0$. Then $f(x) = 0$ for all $x \in E$ and there is nothing to prove.

Choose *a* sequence $\{B_n\}$ $(B = B_1 > B_2 > B_3 > \cdots)$ with $\lim B_n = 0$. Let

$E_i = E\,(B_{i+1} < f \le B_i)$ $(i = 1, 2, 3, \ldots.)$.

and $E_0 = E\,(f = 0)$.

Then $E_0, E_1, E_2, E_3, \ldots.$ are measurable, pairwise disjoint and $E = \bigcup_{i=0}^{\infty} E_i$. Clearly f is integrable on each E_i

So

$$0 = \int_E f d\mu = \sum_{i=0}^{\infty} \int_{E_i} f d\mu = \sum_{i=1}^{\infty} \int_{E_i} f d\mu.$$

This gives that

$$\int_{E_i} f d\mu = 0 \quad (i = 1, 2, 3, \ldots)$$

We have

$$0 = \int_{E_i} f d\mu \ge B_{i+1} \mu E_i$$

So $\quad \mu E_i = 0 \; (i = 1, 2, 3, \ldots)$

and hence $\mu\left(\sum_{i=1}^{\infty} E_i\right) = 0.$

It is clear that

$$E(f > 0) = \bigcup_{i=1}^{\infty} E_i$$

Since $\mu E(f > 0) = 0$, $f(x) = 0$ *a.e. on* E.

Corollary. 5.8.2. Let the functions f and g be bounded on the measurable set E with $\mu E < +\infty$. If $f(x) = g(x)$ *a.e.* on E and f is intgrable on E, then g is also integrable on E and $\int_E f d\mu = \int_E g d\mu$.

Proof : Let $E_1 = E(f = g)$ and $E_2 = E/E_1$ Then $\mu E_2 = 0$. So g is integrable on E_2. Since f is integrable on E_1 and $f(x) = g(x)$ for all $x \in E_1$, g is also integrable on E_1. Hence g is integrable on $E = E_1 \cup E_2$ and

$$\int_E g d\mu = \int_{E_1} g d\mu + \int_{E_2} g du = \int_{E_1} f d\mu + \int_{E_2} f d\mu = \int_E f d\mu$$

Theorem 5.9. Let the function f be bounded and integrable on the set E with $\mu E < +\infty$. Then f is measurable on E.

Proof : We choose *a* sequence $\{P_n\}$ of measurable partitions

$$P_n = \{E_1^{(n)}, E_2^{(n)}, \ldots, E_{r_n}^{(n)}\}$$

of the set E such that

(i) $P_{n+1} \geq P_n$ and (ii) $S_{P_n}(f) - s_{P_n}(f) < \frac{1}{n}$.

Denote by $A_i^{(n)}$ and $B_i^{(n)}$ the glb and lub of f on the set $E_i^{(n)}$. We define the functions g_n and h_n on E as follows.

$$\left.\begin{aligned} g_n(x) &= A_i^{(n)} \\ h_n(x) &= B_i^{(n)} \end{aligned}\right\} \text{for } x \in E_i^{(n)} \ (i = 1, 2, \ldots, p_n)$$

Clearly the functions g_n and h_n are measurable and bounded on E and (1) $g_n \leq f \leq h_n$ (2) $g_n \leq g_{n+1}$ (3) $h_n \geq h_{n+1}$.

So for each $x \in E$,

$$\lim_{n\to\infty} g_n(x) \text{ and } \lim_{n\to\infty} h_n(x) \text{ exist.}$$

For $x \in E$, let

$$g(x) = \lim_{n\to\infty} g_n(x) \text{ and } h(x) = \lim_{n\to\infty} h_n(x).$$

Then g and h are bounded and measurable on E. From (1) we have

$$g_n \le g \le f \le h \le h_n$$

So
$$\int_E g_n d\mu \le \int_E g d\mu \le \int_E h d\mu \le \int_E h_n d\mu$$

or
$$s_{P_n}(f) \le \int_E g d\mu \le \int_E g d\mu \le S_{P_n}(f)$$

or
$$0 \le \int_E (h-g) d\mu \le S_{P_n}(f) - s_{P_n}(f) < \frac{1}{n}.$$

Letting $n \to \infty$ we obtain,

$$\int_E (h-g) d\mu = 0.$$

Since $h(x) - g(x) \ge 0$ for all $x \in E$, by corollary 5.8.1., we get $h(x)-g(x)=0$ *a.e.* on E, that is, $h(x) = g(x)$ *a.e.* on E. Again, since $g(x) \le f(x) \le h(x)$ for all $x \in E$, $f(x) = g(x) = h(x)$ *a.e.* on E.Hence f is measurable on E.

Theorem 5.10. (Lebesgue bounded convergence Theorem)

Let $\{f_n\}$ be *a* sequence of bounded functions measurable on the set E with $\mu E < +\infty$ and let $f_n(x) \to f(x)$ *a. e.* on E. If there is *a* positive number k such that $|f_n(x)| \le k$ *a. e.* on E and f is bounded on E, f is integrable on E and

$$\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu.$$

Proof : Let $A_0 = E(f_n \not\to f)$ and $A_n = E(|f_n| > k)$ $(n = 1, 2, 3, \ldots)$. Then $A_0, A_1, A_2, A_3, \ldots$ are measurable and $\mu A_n = 0$ $(n = 0, 1, 2, 3, \ldots)$.

Write $A = \bigcup_{n=0}^{\infty} A_n$ and $B = E \setminus A$.

We have $\mu A = 0$ and $\mu B = \mu E$.

Clearly f is measurable on E. Also f is bounded on E. So f is integrable on E. Let $M(\ge k)$ be *a* positive number such that $|f(x)| \le M$ for all $x \in E$.

Choose any $\varepsilon > 0$. For each n, let

$$B_n = \bigcap_{\nu=n}^{\infty} B(|f_\nu - f| < \varepsilon).$$

Then the sets $B_1, B_2, B_3, \ldots$ are measurable, $B_1 \subset B_2 \subset B_3 \subset \ldots$ and $B = \bigcup_{n=1}^{\infty} B_n$. So $\mu B_n \to \mu B$ as $n \to \infty$. Determine the positive integer N such that $\mu B_N > \mu B - \varepsilon$.

Take any $n \geq N$. Then

$$\left|\int_E f_n d\mu - \int_E f d\mu\right| = \left|\int_E (f_n - f)\, d\mu\right|$$

$$\leq \int_E |f_n - f\,| d\mu$$

$$= \int_{B_N} |f_n - f\,| d\mu + \int_{E/B_N} |f_n - f| d\mu$$

$$\leq \varepsilon . \mu\, B_N + 2M . \mu(E \mid B_N)$$

$$\leq (\mu E + 2M). \varepsilon$$

This gives that

$$\int_E f d\mu = \lim_{n\to\infty} \int_E \int_n d\mu .$$

Theorem 5.11. Let the functions $f, f_1, f_2, f_3,$ ...be real-valued bounded on the closed interval $[a, b]$ and let they be integrable (R) on $[a, b]$. If $f_n(x) \to f(x)$ for all $x \in [a, b]$ and $\{f_n\}$ is uniformly bounded on $[a, b]$, then

$$(R)\int_a^b f(x)dx = \lim_{n\to\infty} (R)\int_a^b f_n(x)dx .$$

Proof : Let μ denote the usual Lebesgue measure on the real line and let $E = [a, b]$.

Since the functions $f, f_1, f_2, f_3,$ ....are integrable (R) on $[a, b]$, they are integrable (L) also on $[a, b]$ and the corresponding integrals are equal. By Lebesgue bounded convergence Theorem

$$(L) \int_E f d\mu = \lim_{n\to\infty} (L) \int_E f_n\, d\mu$$

or

$$(R) \int_a^b f(x)dx = \lim_{n\to\infty} (R) \int_a^b f_n(x)dx.$$

CHAPTER—VI

LEBESGUE INTEGRAL FOR NON-NEGATIVE FUNCTION

Let $(X, \mathcal{S}, \mu)$ be a complete measure space and let E be a measurable subset of X and f be a non-negative measurable function on E. We extend the definition of Lebesgue Integral of non-negative functions in two stages. First we extend to measurable set with finite measure and then to any measarable set.

(*A*) Suppose that the measure of the set E is finite. For any positive integer N we define the function $[f]_N$ on E as follows.

$$[f]_N(x) = f(x) \text{ if } 0 \le f(x) \le N,$$
$$= N \text{ if } f(x) > \text{N}.$$

Then $[f]_N$ is measurable on the set E and $0 \le [f]_N(x) \le N$ for all $x \in E$. Since $[f]_N(x) \le [f]_{N+1}(x)$ for all $x \in E$, we have

$$\int_E [f]_N du \le \int_E [f]_{N+1} d\mu.$$

This gives that $\lim\limits_{N\to\infty} [f]_N d\mu$ exists which we denote by $\int_E f d\mu$ and call it the Lebesgue Integral of f on E. If $\int_E f d\mu < +\infty$, we say that f is integrable (L) or summable on E.

(*B*) Next, suppose that the measure of the set E is not finite. Denote by Δ_E the class of all measurable subsets of E with finite measure.

For e_1, e_2 in Δ_E let us define $e_1 \le e_2$ *if* $e_1 \subset e_2$. It is easy to see that $(\Delta_E, \le)$ is *a* directed set. For each e in Δ_E, $\int_e f d\mu$ is defined. Clearly $\left\{\int_e f d\mu : e \in \Delta_E\right\}$ is *an* increasing net. So $\lim\limits_{e} \int_e f d\mu$ exists which we denote by $\int_E f d\mu$ and call it the Lebesgue integral of f on E. If $\int_E f d\mu < +\infty$, we say that f is integrable (L) or summable on E. It is clear that

$$\int_E f d\mu = \sup \left\{\int_e f d\mu : e \in \Delta_E\right\}.$$

Theorem 6.1. Let f be measurable and nonnegative on E. If A is a measurable subset of E, then

$$\int_A f d\mu \le \int_E f d\mu.$$

Proof : (I) First suppose that the measure of E is finite. Take any positive integer N. Write $B = E \setminus A$. Then B is measurable and $E = A \cup B$, $A \cap B = \phi$. Since $[f]_N$ is bounded, measurable and non-negative on E, we have

$$\int_E [f]_N d\mu = \int_A [[f]_N d\mu + \int_B [f]_N d\mu.$$

$$\ge \int_A [f]_N d\mu.$$

Letting $N \to \infty$ we obtain

$$\int_A f d\mu \le \int_E f d\mu.$$

(II) Now let the measure of E be infinite. Take any measurable subset e of A with finite measure. Then e is also a measurable subset of E. So

$$\int_e f d\mu \le \int_E f d\mu.$$

Taking supremum over all measurable subsets e of A with finite measure we get

$$\int_A f d\mu \le \int_E f d\mu.$$

Theorem 6.2. Let E be a measurable set with $\mu E = 0$. Then any non-negative function f on E is summable on E and

$$\int_E f d\mu = 0.$$

Proof : Let N be any positive integer. Clearly f is measurable on E and so $[f]_N$ is measurable on E. Since $\mu E = 0$,

$$\int_E [f]_N d\mu = 0.$$

This is true for every positive integer N.

So letting $N \to \infty$ we get

$$\int_E f d\mu = 0.$$

Theorem 6.3. Let E be a measurable set with σ-finite measure. If f is non-negative and summable on E, then f is finite $a.e.$ on E.

Proof : (I) We first suppose that the measure of E is finite. Write

$$A_n = E(0 \le f \le n) \ (n = 1, 2, 3, \ldots).$$

and $A = E \mid \left(\bigcup_{n=1}^{\infty} A_n\right).$

Then $f(x) = +\infty$ for all $x \in A$.

Since f is measurable on E, the sets $A_1, A_2, A_3, \ldots$ are measurable which gives that A is measurable.

Take any positive integer N. Then

$$[f]_N(x) = N \text{ for all } x \in A.$$

So $N.\mu(A) = \int_A [f]_N d\mu \leq \int_E f d\mu \leq k$, where k is some positive number.

or $$0 \leq \mu(A) \leq \frac{K}{N}.$$

Letting $N \to \infty$ we get $\mu(A) = 0$.

Hence f is finite *a.e.* on E.

(II) Now suppose that measure of E is infinite. Since the measure of E is σ-finite, there is *a* sequence $\{E_n\}$ of pairwise disjoint measurable sets such that

$$E = \bigcup_{n=1}^{\infty} E_n \text{ and } \mu(E_n) < +\infty \ (n = 1, 2, \ldots..).$$

For each n, we have

$$\int_{E_n} f d\mu \leq \int_E f du.$$

which gives that f summable on E_n. So by case *I*, f is finite *a.e.* on E_n and hence f is finite *a.e.* on E.

Theorem 6.4. Let the functions f and g be non-negative and measurable on E. If

$f(x) \leq g(x)$ for all $x \in E$, then

$$\int_E f d\mu \leq \int_E g d\mu.$$

Proof : (I) First suppose that μE is finite. Choose any positive integer N. Then

$$[f]_N(x) \leq [g]_N(x) \text{ for all } x \in E.$$

Further $[f]_N$ and $[g]_N$ are measurable and bounded on E. So

$$\int_E [f] d\mu \leq \int_E [g]_N d\mu.$$

Letting $N \to \infty$ we get

$$\int_E f d\mu \leq \int_E g d\mu.$$

(II) Next, suppose that $\mu E = +\infty$. Take any measurable sub set e of E with finite measure. Then by case *I*,

$$\int_e f d\mu \leq \int_e g d\mu.$$

Taking supremum over all measurable subsets e of E with finite measure we get

$$\int_E f d\mu \leq \int_E g d\mu.$$

Theorem 6.5. Let the function f be non-negative and measurable on the set E.

If $\int_E f d\mu = 0$ and the measure of E is σ-finite, the $f(x) = 0$ *a.e.* on E.

Proof : (I) First suppose that μE is finite. Let $A_n = E\left(f \geq \frac{1}{n}\right)$ $(n = 1,2,3, \ldots)$ and $A = \bigcup_{n=1}^{\infty} A_n$. Clearly the sets $A_1, A_2, A_3, \ldots$ are measurable and so A is measurable.

Since $\frac{1}{n} \leq f(x)$ for all $x \in A_n$, we get

$$0 \leq \frac{1}{n}\, \mu(A_n) \leq \int_{A_n} f d\mu \leq \int_E f d\mu \leq \int_E f d\mu = 0.$$

This gives that $\mu(A_n) = 0$ for $n = 1, 2, 3, \ldots$.and so $\mu\, A = 0$. since $A = E(f > 0)$, it follows that $f(x) = 0$ *a.e.* on E.

(II) Next, sppose that $\mu E = +\infty$. Since the measure of E is σ-finite, there is a sequence $\{E_n\}$ of pairwise disjoint measurable sets such that $E = \bigcup_{n=1}^{\infty} E_n$ and $\mu\, E_n < +\infty$ $(n = 1, 2, 3, \ldots)$.

Since $0 \leq \int_{E_n} f d\mu \leq \int_E f d\mu = 0$, we get

$$\int_{E_n} d\mu = 0 \ (n = 1, 2, 3, \ldots).$$

By case *I*, $f(x) = 0$ *a.e.* on E_n and so $f(x) = 0$ *a.e.* on E.

Theorem 6.6. Let the function f and g be non-negative and measurable on the set E. Then

$$\int_E (f + g) d\mu = \int_E f d\mu + \int_E g d\mu.$$

Proof : (I) First we suppose that μE is finite. Take any positive integer N. Then

$$[f]_N(x) + [g]_N(x) \leq f(x) + g(x) \text{ for } x \in E.$$

So
$$\int_E [f]_N d\mu + \int_E [g]_N dx \leq \int_E (f + g) d\mu$$

Let $N \to \infty$ we have

$$\int_E f d\mu + \int_E g d\mu \leq \int_E (f+g) d\mu. \qquad \dots (6.1)$$

Again for any positive integer N,

$$[f+g]_N(x) \leq [f]_N(x) + [g]_N(x) \text{ for } x \in E.$$

So $$\int_E [f+g]_N \, d\mu \leq \int_E [f]_N d\mu + \int_E [g]_N \, d\mu.$$

Letting $N \to \infty$ we get

$$\int_E [f+g] d\mu \leq \int_E f d\mu + \int_E g d\mu \qquad \dots \quad \dots \quad (6.2)$$

From (6.1) and (6.2) we obtain

$$\int_E (f+g) \, d\mu = \int_E f d\mu + \int_E g d\mu.$$

(II) *Next,* let $\mu E = +\infty$. Take any measurable subset e of E with finite measure. By case I,

$$\int_e (f+g) d\mu = \int_e f d\mu + \int_e g d\mu$$

This gives that

$$\int_E (f+g) d\mu = \int_E f d\mu + \int_E g d\mu.$$

Theorem 6.8. (Fatou's Lemma).

Let $\{f_n\}$ be *a* sequence of non-negative functions measurable on the set E. If $f_n(x) \to f(x)$ *a.e.* on E, then

$$\int_E f d\mu \leq \sup\left\{\int_E f_n d\mu\right\}.$$

Proof : Clearly the function f is measurable on E. We may suppose that $f(x) \geq 0$ for all $x \in E$. Let F denote the set of all points x in E such that $f_n(x) \to f(x)$. Then $\mu\,(E \setminus F) = 0$.

(I) First suppose that μE is finite. Choose any positive integer N. Let $x \in F$. We show that $[f_n]_N(x) \to [f]_N(x)$ as $n \to \infty$. Suppose that $f(x) < N$. There is *a* positive integer n_0 such that $f_n(x) < N$ for all $n \geq n_0$.

So $[f_n]_N(x) = f_n(x)$ for all $n \geq n_0$.

Also $[f]_N(x) = f(x)$. Hence

$$[f_n]_N(x) \to [f]_N(x) \text{ as } n \to \infty.$$

Now let $f(x) > N$. Then there is *a* positive integer n_0 such that $f_n(x) > N$

for all $n \geq n_0$. So $[f_n]_N(x) = N = [f]_N(x)$ for all $n \geq n_0$. Hence $[f_n]_N(x) \rightarrow [f]_N(x)$ as $n \rightarrow \infty$.

Lastly, let $f(x) = N$. Choose any ε with $0 < \varepsilon < 1$. We can find *a* positive integer n_0 such that

$N - \varepsilon < f_n(x) < N + \varepsilon$ for all $n \geq n_0$

This gives that

$$N - \varepsilon < [f_n]_N(x) \leq N \text{ when } n \geq n_0.$$

Hence $[f_n]_N(x) \rightarrow N = [f]_N(x)$ as $n \rightarrow \infty$.

Since $0 \leq [f_n]_N(x) \leq N$ for all $x \in E$ by Lebesgue bounded convergence Theorem

$$\int_E [f]_N \, d\mu = \lim_{n\to\infty} \int_E [f_n]_N \, d\mu \quad \ldots \quad (6.3)$$

Now for any n,

$$\int_E [f_n]_N \, d\mu \leq \int_E f_n d\mu \leq \sup\left\{\int_E f_n d\mu\right\} \quad \ldots (6.4)$$

From (6.3) and (6.4) we get

$$\int_E [f]_N \, d\mu \leq \sup\left\{\int_E f_n d\mu\right\} \quad \ldots \quad (6.5)$$

Letting $N \rightarrow \infty$ in (6.5) we obtain

$$\int_E f d\mu \leq \sup\left\{\int_E f_n d\mu\right\}.$$

(II) Next, suppose that $\mu E = +\infty$. Let e be any measurable subset of E with finite measure. Then by case I,

$$\int_e f d\mu \leq \sup\left\{\int_e f_n d\mu\right\} \quad \ldots \quad \ldots \quad (6.6)$$

Since $\int_e f_n d\mu \leq \int_E f_n d\mu \qquad (n = 1, 2, 3, \ldots)$

$$\text{Sup}\left\{\int_e f_n d\mu\right\} \leq \sup\left\{\int_E f_n d\mu\right\} \quad \ldots \quad \ldots \quad (6.7)$$

From (6.6) and (6.7) we get

$$\int_e f d\mu \leq \text{Sup}\left\{\int_E f_n d\mu\right\}.$$

This gives that

$$\int_E f d\mu \leq \text{Sup}\left\{\int_E f_n d\mu\right\}.$$

Corollary 6.8.1. Let $\{f_n\}$ be a sequence of nonnegative functions measurable on the set E. If $\lim_{n\to\infty} \int_E f_n d\mu$ exists and $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. on E, then

$$\int_E f d\mu \leq \lim_{n\to\infty} \int_E f_n d\mu.$$

Proof : If $\lim_{n\to\infty} \int_E f_n d\mu = +\infty$, then the result is trivial. So we suppose that

$$\lim_{n\to\infty} \int_E f_n = l \text{ (finite)}.$$

Choose any $\varepsilon > 0$. Then there is a positive integer m such that

$$\int_E f_n \, d\mu < l + \varepsilon \text{ for all } n \geq m.$$

Consider the sequence $\{f_n\}_{n=m}^{\infty}$. By Fatou's Lemma, we get

$$\int_E f d\mu \leq \sup_{n \geq m} \left\{\int_E f_n d\mu\right\} \leq l + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\int_E f d\mu \leq \lim_{n\to\infty} \int_E f_n d\mu.$$

Corollary 6.8.2. Let $\{f_n\}$ be a sequence of non-negative functions measurable on the set E.

If $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. on E, then $\int_E f d\mu \leq \lim_{n\to\infty} \inf \int_E f_n d\mu$.

Proof : We can choose a sequence $\{n_k\} (n_1 < n_2 < n_3 < \ldots)$ of positive integers such that

$$\lim_{k\to\infty} \int_E f_{n_k} d\mu = \lim_{n\to\infty} \inf \int_E f_n d\mu.$$

Now consider the sequence $\{f_{n_k}\}$. Since $\{f_{n_k}(x)\}$ converges to $f(x)$ a.e. on E by corollary 6.8.1. we get

$$\int_E f d\mu \leq \lim_{k\to\infty} \int_E f_{n_k} d\mu = \lim_{n\to\infty} \inf \int_E f_n d\mu.$$

Theorem 6.9. (Lebesgue monotone convergence Theorem).

Let $\{f_n\}$ be *a* sequence of non-negative functions measurable on the set E. If for every x in E, $f_1(x) \le f_3(x) \le \ldots$ and $f(x) = \lim_{n\to\infty} f_n(x)$, then

$$\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu .$$

Proof : Clearly f is non-negative and measurable on E.

Since $f_1(x) \le f_2(x) \le f_3(x) \le \ldots \le f(x)$ on E,

$$\int_E f_1 d\mu \le \int_E f_2 d\mu \le \int_E f_3 d\mu \le \cdots \le \int_E f d\mu .$$

This gives that

$$\lim_{n\to\infty} \int_E f_n d\mu \le \int_E f d\mu . \qquad \ldots \qquad \ldots \qquad (6.8)$$

To prove the reverse inequality we proceed as follows.

Let e be any measurable subset of E with finite measure. Choose any positive integer N and let $x \in e$. If $f(x) \le N$, then $f_n(x) \le N$ for all n.

So $[f_n]_N(x) = f_n(x)$ and $[f]_N(x) = f(x)$. This gives that $[f_n]_N(x) \to [f]_N(x)$ as $n \to \infty$.

Next, suppose that $f(x) > N$. There is *a* posinttive integer n_0 such that

$$f_n(x) > N \text{ for all } n \ge n_0.$$

So $[f_n]_N(x) = N = [f]_N(x)$ for all $n \ge n_0$ which gives that

$$[f_n]_N(x) \to [f]_N(x) \text{ as } n \to \infty.$$

By Lebesgue bounded convergence Theorem

$$\int_e [f]_N d\mu = \lim_{n\to\infty} \int_e [f_n]_N d\mu \qquad \ldots \qquad \ldots \qquad (6.9)$$

Since $\int_e [f_n]_N d\mu \le \int_e f_n d\mu \le \int_E f_n d\mu ,$

$$\lim_{n\to\infty} \int_e [f_n]_N d\mu \le \lim_{n\to\infty} \int_E f_n d\mu \qquad \ldots \qquad \ldots \qquad (6.10)$$

From (6.9) and (6.10) we get

$$\int_e [f]_N d\mu \le \lim_{n\to\infty} \int_E f_n d\mu .$$

Letting $N \to \infty$ we have

$$\int_e f d\mu \leq \lim_{n\to\infty} \int_E f_n d\mu .$$

This gives that

$$\int_E f d\mu \leq \lim_{n\to\infty} \int_E f_n d\mu . \quad \dots \quad \dots \quad (6.11)$$

From (6.8) and (6.11) we obtain

$$\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu .$$

Example 6.1. Prove that Fatou's Lemma and Lebesgue monotone convergence Theorem are equivalent.

Solution.

(I) First we deduce Lebesgue monotone convergence Theorem from Fatou's Lemma.

Let $\{f_n\}$ be *a* sequence of non-negative functions measurable on the set E and for $x \in E$, let $f_n(x) \leq f_{n+1}(x)$ $(n=1, 2, 3, \dots)$; and $f(x) = \lim f_n(x)$.

Since $\quad f_1(x) \leq f_2(x) \leq f_3(x) \leq f_3(x) \leq \dots \leq f(x)$ on E,

$$\int_E f_1 d\mu \leq \int_E f_2 d\mu \leq \cdots \leq \int_E f d\mu .$$

So

$$\lim_{n\to\infty} \int_E f_n d\mu \leq \int_E f d\mu \quad \dots \quad \dots \quad (6.12)$$

By Fatou's Lemma

$$\int_E f d\mu \leq sup\left\{\int_E f_n d\mu\right\} = \lim_{n\to\infty} \int_E f_n d\mu \quad \dots \quad \dots \quad (6.13)$$

From (6.12) and (6.13) we get

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu .$$

(II) Now let $\{f_n\}$ be *a* sequence of non-negative functions measurable on E, and $\lim\limits_{n\to\infty} f_n(x) = f(x)$ *a.e. on* E.

For any $x \in E$, let

$$g_n(x) = \inf\{ f_\nu(x) : \nu \geq n\}.$$

Then $g_1(x) \leq g_2(x) \leq g_3(x) \leq \ldots$ on E

and $$f(x) = \lim_{n\to\infty} g_n(x) \text{ a.e. on} E.$$

By Lebesque monotone convergence Theorem

$$\int_E f d\mu = \lim_{n\to\infty} \int_E g_n d\mu \quad \ldots \quad \ldots \quad (6.14)$$

Since $g_n(x) \leq f_n(x)$ for all $x \in E$,

$$\int_E g_n d\mu \leq \int_E f_n d\mu \leq sup\left\{\int_E f_n d\mu\right\}.$$

This gives that

$$\lim_{n\to\infty} \int_E g_n d\mu \leq sup\left\{\int_E f_n d\mu\right\} \quad \ldots \quad \ldots \quad (6.15)$$

From (6.14) and (6.15),

$$\int_E f d\mu \leq sup\left\{\int_E f_n d\mu\right\}.$$

Corollary 6.9.1. Let $\{u_n\}$ be *a* sequence of non-negative functions measurable on the set E.

If $f(x) = \sum_{n=1}^{\infty} u_n(x)$ for $x \in E$, then

$$\int_E f d\mu = \sum_{n=1}^{\infty} \int_E u_n d\mu .$$

Proof : Let $f_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x)$ for all $x \in E$. Then

$f_1(x) \leq f_2(x) \leq f_3(x) \leq \cdots \leq f(x)$ and $f(x) = \lim_{n\to\infty} f_n(x)$ on E.

Clearly $f, f_1, f_2, \cdots, f_n, \cdots$ are all measurable on E. By Lebesgue monotone convergence Theorem,

$$\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu$$

$$= \lim_{n\to\infty} \sum_{\nu=1}^{n} \int_E u_\nu d\mu$$

$$= \sum_{\nu=1}^{\infty} \int_E u_\nu d\mu .$$

Theorem 6.10. (Countable additivity of integral).

Let the function f be non-negative and measurable on the set E; and let E can be expressed as the union of a countable class $\{E_i\}$ of pairwise disjoint measurable sets E_i. Then

$$\int_E f d\mu = \sum_i \int_{E_i} f_i d\mu . \quad \ldots \quad \ldots \quad (8.16)$$

Proof : If $\int_{E_i} fdu = +\infty$ for some i, then clearly $\int_E fdu = +\infty$ and equality (6.16) holds. So we suppose that $\int_{E_i} fdu$ is finite for all i.

Take any measurable $e \subset E$ with finite measure and let $e_i = E_i \cap e$. Choose any positive integer N. Then we have

$$\int_e [f]_N d\mu = \sum_i \int_{e_i} [f]_N d\mu \leq \sum_i \int_{e_i} fd\mu$$

Letting $N \to \infty$ we get

$$\int_e fd\mu \leq \sum_i \int_{e_i} fd\mu \leq \sum_i \int_{E_i} fd\mu$$

This gives that

$$\int_E fd\mu \leq \sum_i \int_{E_i} fd\mu . \quad \ldots (6.17)$$

To prove the reverse inequality we proceed as follows.

Take any positive integer m. Choose any $\varepsilon > 0$. For each i $(1 \leq i \leq m)$ we can find a measurable set $e_i \subset E_i$ with finite measure such that

$$\int_{e_i} fd\mu > \int_{E_i} fd\mu - \varepsilon/_i \quad \ldots \quad \ldots \quad (6.18)$$

Write $e = e_1 \cup e_2 \cup \ldots \cup e_m$. Then e is a measurable subset of E with finite measure. For any positive integer N,

$$\sum_{i=1}^{m} \int_{e_i} [f]_N d\mu = \int_e [f]_N d\mu \leq \int_e fd\mu \leq \int_E fd\mu .$$

Letting $N \to \infty$ we get

$$\sum_{i=1}^{m} \int_{e_i} f d\mu \le \int_E f d\mu$$

Using (6.18) we have

$$\sum_{i=1}^{m} \int_{E_i} f d\mu < \int_E f d\mu + \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary we obtain

$$\sum_{i=1}^{m} \int_{E_i} f d\mu \le \int_E f d\mu . \qquad \text{... ... (6.19)}$$

The inequality (6.19) holds for every positive integer m. So

$$\sum_{i} \int_{E_i} f d\mu \le \int_E f d\mu . \qquad \text{... ... (6.20)}$$

From (6.17) and (6.20) we have

$$\int_E f d\mu = \sum_{i} \int_{E_i} f d\mu .$$

CHAPTER—VII

GENERAL LEBESGUE INTEGRAL

Let $(X, \mathcal{S}, \mu)$ be *a* complete measure space and let E be *a* measurable subset of X and $f : E \to \Omega$ be measurable, where Ω denotes the set of all real numbers. We define the functions f_+ and f_- on E as follows.

$$f_+(x) = \begin{cases} f(x) \text{ if } f(x) \geq 0 \\ 0 \text{ if } f(x) < 0 \end{cases} \quad \text{and } f_-(x) = \begin{cases} -f(x) \text{ if } f(x) \leq 0 \\ 0 \text{ if } f(x) > 0. \end{cases}$$

Then the functions f_+ and f_- are measurable on E and
$f(x) = f_+(x) - f_-(x)$ for all $x \in E$.
If at least one of the integrals

$$\int_E f_+ d\mu \text{ and } \int_E f_- d\mu$$

is finite, then the difference

$$\int_E f_- d\mu - \int_E f_- d\mu$$

is called the Lebesgue Integral of f on E and is denoted by $\int_E f d\mu$. If each of f_+ and f_- is summable on E, then f is said to be summable on E or integrable on E.

Theorem 7.1. Let f be *a* measurable function on the set E. Then f is summable on E if and only if $|f|$ is summable on E. If f is summable on E, then

$$\left|\int_E f d\mu\right| \leq \int_E |f| d\mu.$$

Proof : It is easy to see that

$$|f(x)| = f_+(x) + f_-(x) \text{ for all } x \in E \qquad \dots \quad \dots \quad (7.1)$$

First suppose that f is summable on E. Then the integrals $\int_E f_+ d\mu$ and $\int_E f_- d\mu$ are finite. From (7.1) we get

$$\int_E |f| d\mu = \int_E f_+ d\mu + \int_E f_- d\mu. \qquad \dots \quad \dots \quad (7.2)$$

This gives that $|f|$ is summable on E.

Next, suppose that $|f|$ is summable on E. Since $f_+(x) \leq |f(x)|$ and $f_-(x) \leq |f(x)|$ for $x \in E$, it follows that f_+ and f_- are summable on E. So the function f is summable on E.

Now suppose that f is summable on E. We have

$$\int_E f d\mu = \int_E f_+ d\mu - \int_E f_- d\mu \leq \int_E (f_+ + f_-) d\mu$$

$$\leq \int_E |f| d\mu.$$

Again

$$\int_E f d\mu \geq -\int_E f_- d\mu - \int_E f_- d\mu = -\int_E (f_+ + f_-) d\mu$$

$$\geq -\int_E |f| d\mu.$$

Hence

$$\left|\int_E f d\mu\right| \leq \int_E |f| d\mu.$$

Corollary 7.1.1 If f is summable on E and if A is a measurable subset of E, then f is summable on A.

Since $\int_A |f| d\mu \leq \int_E |f| d\mu,$

the result follows.

Corollary 7.1.2. Let f and g be measurable on the set E and $|f(x)| \leq g(x)$ for $x \in E$. If g is summable on E, then f is also summable on E.

Since $\int_E |f| d\mu \leq \int_E g d\mu$ and g is summable on E, $|f|$ is summable on E. Hence f is summable on E.

Corollary 7.1.3. Let the function f be summable on E. Then f is finite *a.e.* on E.

Proof : Since f is summable on E, $|f|$ is also summable on E. By Theorem 6.3, $|f|$ is finite *a. e.* on E, that is, f is finite *a. e.* on E.

Theorem 7.2. Let the function f be summable on the set E. If E can be expressed as the union of *a* countable family $\{E_i\}$ of pairwise dijoint measurable sets, then

$$\int_E f d\mu = \sum_i \int_{E_i} f d\mu$$

Proof : Since each E_i is *a* measurable subset of E and f is summable on E, it follows that f_+ and f_- are summable on each E_i. Also we have

$$\int_E f_+ d\mu = \sum_i \int_{E_i} f_+ d\mu \text{ and } \int_E f_- d\mu = \sum_i \int_{E_i} f_- d\mu.$$

So

$$\int_E f d\mu = \int_E f_+ d\mu - \int_E f_- d\mu$$

$$= \sum_i \int_{E_i} f_+ d\mu - \sum_i \int_{E_i} f_- d\mu$$

$$= \sum_i \int_{E_i} (f_+ - f_-) d\mu = \sum_i \int_{E_i} f d\mu.$$

Corollary 7.2.1. Let E be the union of a countable fanily $\{E_i\}$ of pairwise disjoint sets E_i and let f be summable on each E_i. If $\sum_i \int_{E_i} |f| d\mu < +\infty$, then f is summable on E and

$$\int_E f d\mu = \sum_i \int_{E_i} f \, d\mu .$$

Proof : We have

$$\int_E |f| d\mu = \sum_i \int_{E_i} |f| d\mu .$$

Since $\sum_i \int_{E_i} |f| d\mu < +\infty$, it follows that $|f|$ is summable on E.

Hence

$$\int_E f d\mu = \sum_i \int_{E_i} f d\mu.$$

Theorem 7.3. Let f be summable on the set E and k be any real number. Then kf is summable on E and

$$\int_E (kf) d\mu = k \int_E f d\mu.$$

Proof : If $k = 0$, the result is trivial. Suppose that $k > 0$. Then

$$(kf)_+ = kf_+ \text{ and } (kf)_- = kf_-.$$

This gives that $(kf)_+$ and $(kf)_-$ are summable on E and so kf is summable on E. We have

$$\int_E (kf) d\mu = \int_E (kf)_+ d\mu - \int_E (kf)_- d\mu$$

$$= \int_E kf_+ d\mu - \int_E kf_- d\mu$$

$$= k \int_E (f_+ - f_-) d\mu = k \int_E f d\mu .$$

Next, let $k < 0$. Then

$$(kf)_+ = (-k) f_- \text{ and } (kf)_- = (-k) f_+.$$

This gives that $(kf)_+$ and $(kf)_-$ are summable on E. Hence kf is summable on E. We have

$$\int_E (kf) d\mu = \int_E (kf)_+ d\mu - \int_E (kf)_- d\mu$$

$$= \int_E (-k) f_- d\mu - \int_E (-k) f_+ d\mu$$

$$= k \int_E (f_+ - f_-) d\mu = k \int_E f d\mu$$

Theorem 7.4. Let the functions f and g be summable on the set E. Then $f + g$ is also summable on E and

$$\int_E (f+g)d\mu = \int_E f d\mu + \int_E g d\mu.$$

Proof : Let $\phi(x) = f(x) + g(x)$ for all $x \in E$.

Then $$|\phi(x)| \leq |f(x)| + |g(x)| \text{ for all } x \in E.$$

Since f and g are summable on E, it follows that ϕ is summable on E. We now define the sets E_1, E_2, E_3, E_4, E_5, E_6 as follows.

$E_1 = E\ (f \geq 0,\ g \geq 0)$, $E_4 = E(f < 0,\ g < 0)$

$E_2 = E(f \geq 0,\ g < 0,\ \phi \geq 0)$ $E_5 = E(f \geq 0,\ g < 0,\ \phi < 0)$

$E_3 = E(f < 0,\ g \geq 0,\ \phi \geq 0)$, $E_6 = E(f < 0,\ g \geq 0,\ \phi < 0)$.

Clearly the sets E_1, E_2, E_3, E_4, E_5, E_6 are measurable and pairwise disjoint and

$$E = E_1 \cup E_2 \cup \cdots \cup E_6.$$

We have

$$\int_E \phi d\mu = \sum_{i=1}^{6} \int_{E_i} \phi d\mu,$$

$$\int_E f d\mu = \sum_{i=1}^{6} \int_{E_i} f d\mu, \quad \int_E g d\mu = \sum_{i=1}^{6} \int_{E_i} g d\mu.$$

We now show that

$$\int_{E_i} \phi d\mu = \int_{E_i} f d\mu + \int_{E_i} g d\mu \qquad (i = 1, 2, 3, 4, 5, 6) \ldots(7.3)$$

We prove the relation (7.3) for $i = 2$. The proofs in other cases are similar. From the relation $\phi = f + g$ we have $f = \phi + (-g)$. On the set E_2, ϕ and $(-g)$ are non-negative. So

$$\int_{E_2} f d\mu = \int_{E_2} \phi d\mu + \int_{E_2} (-g) d\mu$$

or $$\int_{E_2} \phi d\mu = \int_{E_2} f d\mu + \int_{E_2} g d\mu.$$

This proves the theorem.

Theorem 7.5. Let the function f be summable on the set E. Then for every $\varepsilon > 0$ there is *a* $\delta > 0$ such that for any measurable set $e \subset E$

$$\left|\int_e f du\right| < \varepsilon \text{ whenever } \mu e < \delta.$$

Proof : First suppose that $\mu\ E < +\infty$. Choose any $\varepsilon > 0$. Since f is summabl‿ on E, so is $|f|$. There is *a* positive integer N such that

$$\int_E |f| d\mu - \int_E [|f|]_N d\mu < \frac{1}{2}\varepsilon \qquad \ldots \quad \ldots \quad (7.4)$$

Take $\delta = \dfrac{\varepsilon}{2N}$. Let e be any measurable subset of E with $\mu\, e < \delta$. Then

$$\left|\int_e f d\mu\right| \le \int_e |f|\, d\mu = \int_e |f| d\mu - \int_e [|f|]_N\, d\mu + \int_e [|f|]_N\, d\mu$$

$$\le \int_e \{|f| - [|f|]_N\} d\mu + N.\mu e$$

$$\le \int_E \{|f| - [|f|]_N\} d\mu + \frac{1}{2}\varepsilon$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \qquad \text{[using (74)]}$$

Now let $\mu E = +\infty$. Then there is *a* measurable subset E_0 of E with $\mu E_0 < +\infty$ such that

$$\int_{E_0} |f| d\mu > \int_E |f| d\mu - \frac{1}{2}\varepsilon.$$

Since $$\int_E |f| d\mu = \int_{E_0} |f| d\mu + \int_{E \setminus E_0} f d\mu,$$

we get

$$\int_{E/E_0} f d\mu < \frac{1}{2}\varepsilon. \qquad \ldots \qquad \ldots \qquad (7.5)$$

By case *I* there is *a* $\delta > 0$ such that for any measurable set $e \subset E_0$ with $\mu e < \delta$,

$$\int_e |f| d\mu < \frac{1}{2}\varepsilon. \qquad \ldots \qquad \ldots \qquad (7.6)$$

Take any measurable set $e \subset E$ with $\mu e < \delta$.
Write $e_1 = E_0 \cap e$ and $e_2 = (E \setminus E_0) \cap e$.
Then $\mu e_1 \le \mu\, e < \delta$. So we get

$$\left|\int_e f d\mu\right| \le \int_e |f| d\mu = \int_{e_1} |f| d\mu + \int_{e_2} |f| d\mu$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \text{ [By (7.5) and (7.6)]}$$

Corollary 7.5.1. Let μ be *a* Lebesgue-stieltjes measure on the real line with $\mu(\{x\}) = 0$ for every real number x. If f is summable on the interval $[a, b]$ and if

$$F(x) = \int_a^x f(t) dt \qquad (a \le x \le b)$$

then F is continuous on $[a, b]$

Theorem 7.6. (Lebesgue dominated convergence Theorem).

Let $\{f_n\}$ be *a* sequence of functions summable on the set E and let $\lim f_n(x) = f(x)$ almost everywhere on E. If there exists *a* function g summable on E such that $|f_n(x)| \leq g(x)$ *a.e.* on E (n = 1, 2, 3, ...), then f is summable on E and

$$\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu .$$

Proof : (I) First we suppose that $\mu E < +\infty$. Let $E_0 = E\,(f_n \nrightarrow f)$ and $E_n = E\,(|f_n| > g)$ (n =1, 2, 3,.....). Then $\mu\, E_n = 0$ (n = 0, 1, 2, 3,...) and so $\mu\left(\bigcup_{n=0}^{\infty} E_n\right) = 0$. Write $A = \bigcup_{n=0}^{\infty} E_n$ and $B = E \setminus A$.

Then $f_n(x) \to f(x)$ and $|f_n(x)| \leq g(x)$ for all $x \in B$. Clearly f is measurable on E and $|f(x)| \leq g(x)$ for all $x \in B$. This gives that f is summable on B and so on E.

Choose any $\varepsilon > 0$. Define the sets B_1, B_2, B_3 ... as follows.

$$B_n = \bigcap_{k=n}^{\infty} B(|f_k - f| < \varepsilon).$$

Then the sets B_1, B_2, B_3,....are measarable, $B_1 \subset B_2 \subset B_3 \subset \ldots$ and

$$B = \bigcup_{n=1}^{\infty} B_n.$$

So $\mu B_n \to \mu\, B$ as $n \to \infty$.

Since g is summable on the set E we can find *a* positive number δ such that for any measurable set $e \subset E$ with $\mu\, e < \delta$

$$\int_e g d\mu < \varepsilon. \qquad \ldots \qquad \ldots \qquad (7.7)$$

We now choose positive integer N such that

$$\mu\, B_n > \mu\, B - \delta \text{ when } n \geq N.$$

Write $e = B \mid B_N$. Take any $n \geq N$.

Then

$$\left|\int_E f d\mu - \int_E f_n d\mu\right| \leq \int_E |f - f_n| d\mu$$

$$= \int_{B_N} |f - f_n| d\mu + \int_e |f - f_n| d\mu$$

$$< \varepsilon.\ \mu\, B_N + 2 \int_e g d\mu$$

$$< \varepsilon.\ \mu E + 2\varepsilon \ [By\ (7.7)]$$

Hence

$$\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu .$$

(II) Next, suppose that $\mu E = +\infty$.

Choose any $\varepsilon > 0$. Since g is summable on E there is *a* measurable subset E_0 of E with $\mu E_0 < +\infty$ such that

$$\int_{E_0} g d\mu > \int_E g d\mu - \frac{1}{3}\varepsilon. \qquad \ldots \qquad (7.8)$$

We have

$$\int_E g d\mu = \int_{E_0} g d\mu + \int_{E \setminus E_0} g d\mu.$$

So using (7.8) we get

$$\int_{E \setminus E_0} g d\mu < \frac{1}{3}\varepsilon \qquad \ldots \qquad \ldots \qquad (7.9)$$

By case(I)

$$\int_{E_0} f d\mu = \lim_{n \to \infty} \int_{E_0} f_n d\mu$$

So there is *a* positive integer N such that for all $n \geq N$,

$$\left| \int_{E_0} f d\mu - \int_{E_0} f_n d\mu \right| < \frac{1}{3}\varepsilon. \qquad \ldots \qquad (7.10)$$

Take any $n \geq N$. We have

$$\left| \int_E f d\mu - \int_E f_n d\mu \right| = \left| \int_{E_0} f d\mu - \int_{E_0} f_n d\mu \right| + \int_{E \setminus E_0} |f - f_n| d\mu$$

$$< \frac{1}{3}\varepsilon + 2\int_{E \setminus E_0} g d\mu$$

$$< \frac{1}{3}\varepsilon + \frac{2}{3}\varepsilon = \varepsilon \text{ [using (7.9) and (7.10)]}$$

Hence

$$\int_E f d\mu = \lim_{n \to \infty} \int_E f_n d\mu.$$

Theorem 7.7. Let $\{f_n\}$ be *a* sequence of functions summable on the set E. If the series $\sum_{n=1}^{\infty} \int_E |f_n| d\mu$ is convergent, then $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely *a.e.* on E to *a* function $f(x)$ summable on E and

$$\int_E f d\mu = \sum_{n=1}^{\infty} \int_E f_n d\mu.$$

Proof : We define the functions $g_1, g_2, g_3, \ldots$ on E as follows. For $x \in E$,

$$g_n(x) = |f_1(x)| + |f_2(x)| + \cdots + |f_n(x)| \quad (n = 1, 2, 3, \ldots).$$

Then the functions $g_1, g_2, g_3 \ldots$ are measurable on E, non-negative and

$$g_1(x) \leq g_2(x) \leq g_3(x) \leq \ldots \text{ for all } n \in E.$$

Write $g(x) = \lim\limits_{n\to\infty} g_n(x)$.

By Lebesgue monotone convergence Theorem

$$\int_E g\,d\mu = \lim_{n\to\infty} \int_E g_n\,d\mu = \lim_{n\to\infty} \sum_{k=1}^{n} \int_E |f_k|\,d\mu$$

$$= \sum_{k=1}^{\infty} \int_E |f_k|\,d\mu < +\infty.$$

This gives that g is summable on E; and so g is finite *a.e.* on E. Since

$$\sum_{k=1}^{\infty} |f_k(x)| \leq g(x) \text{ for all } x \in E,$$

it follows that $\sum\limits_{k=1}^{\infty} f_k(x)$ converges absolutely *a.e.* on E.

Let $f(x) = \sum\limits_{k=1}^{\infty} f_k(x)$ when the series converges and at other points of E, $f(x)$ be defined arbitrarily.

Since $|f(x)| \leq g(x)$ *a.e.* on E, f is summable on E.

Write $h_n(x) = \sum\limits_{k=1}^{n} f_k(x)$ $(n = 1, 2, 3, \ldots)$.

Then $|h_n(x)| \leq g_n(x) \leq g(x)$ *a.e.* on E, and $h_n(x) \to f(x)$ *a.e.* on E.
By Lebesgue dominated convergence Theorem

$$\int_E f\,d\mu = \lim_{n\to\infty} \int_E h_n\,d\mu = \lim_{n\to\infty} \sum_{k=1}^{n} \int_E f_k\,d\mu$$

$$= \sum_{k=1}^{\infty} \int_E f_k\,d\mu,$$

Theorem 7.8. (First Mean value Theorem).

If the function f is bounded and measurable on E and g is summable on

E, then there exists a real number λ lying between the glb and lub of f on E such that

$$\int_E f|g|d\mu = \lambda\int_E |g|d\mu.$$

Proof : Let m and M denote the glb and lub respectively of f on the set E. Then for all $x \in E$,

$$m|g(x)| \leq f(x)\ |g(x)| \leq M|g(x)|$$

So we have

$$m\int_E |g|d\mu \leq \int_E f|g|d\mu \leq M\int_E |g|d\mu \qquad \ldots \quad (7.11)$$

If $\int_E |g|d\mu = 0$, then clearly $\int_E f|g|d\mu = 0$;

and so $\int_E f|g|d\mu = \lambda\int_E |g|d\mu$ for any λ with $m \leq \lambda \leq M$.

Suppose that $\int_E |g|d\mu > 0$. Then from (7.11) we have

$$\int_E f|g|d\mu = \lambda\int_E |g|d\mu,$$

where $m \leq \lambda \leq M$.

Lemma 7.1. (Abel's Lemma).

Let $\{a_1, a_2, \ldots, a_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ be two sets of real numbers with
(i) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq 0$.
(ii) $R \leq v_1 + v_2 + \cdots + v_p \leq K$ $(p = 1, 2, \ldots, n)$.
Then

$$a_1R \leq a_1v_1 + a_2v_2 + \cdots a_nv_n \leq a_1\ K.$$

Proof : Let $w = a_1v_1 + a_2v_2 + \cdots + a_nv_n$

and $s_p = v_1 + v_2 - \cdots + v_p$ $(p = 1, 2, \ldots, n)$.

Then

$$w = a_1s_1 + a_2(s_2 - s_1) + a_3(s_3 - s_2) + \cdots + a_n(s_n - s_{n-1})$$
$$= (a_1 - a_2)\,s_1 + (a_2 - a_3)\,s_2 + \cdots + (a_{n-1} - a_n)s_{n-1} + a_ns_n.$$

So

$$w \geq [(a_1 - a_2) + (a_2 - a_3) + \cdots + (a_{n-1} - a_n) + a_n]\,R = a_1R.$$

and

$$w \leq [(a_1 - a_2) + (a_2 - a_3) + \cdots + (a_{n-1} - a_n) + a_n]K = a_1\ K.$$

Hence

$$a_1\ R \leq w \leq a_1\ K.$$

Theorem. 7.9. (Second Mean value Theorem).

Let μ be a Lebesgue–stieltjes measure on the real lime with $\mu(\{x\}) = 0$ for every real number x. If F is a monotone function on $[a, b]$, then there is a point ξ in $[a, b]$ such that

$$\int_a^b Ffd\mu = F(a)\int_a^\xi fd\mu + F(b)\int_\xi^b fd\mu.$$

Proof : If F is constant, the result is trivial. So we suppose that F *is not* a constant on $[a,b]$. For definiteness we suppose that F is increasing on $[a,b]$. Choose any $\varepsilon > 0$. Then we can find *a* point x_1 in (a, b) such that

$$\int_a^{x_1} |f| d\mu < \varepsilon .$$

Let $\psi(x) = F(b) - F(x)$ for $x \in [a, b]$.

Then ψ is decreasing and non-negative on $[a, b]$. By theorem 1.25 there is *a* subdivision $x_1 < x_2 < x_3 < \cdots < x_n = b$ of $[x_1, b]$ such that for all $x \in (x_i, x_{i+1})$ $(i = 1,2, \cdots, n-1)$

$$\psi\ (x_i+ 0) - \psi(x) < \varepsilon.$$

We now define the function g on $[a, b]$ as follows.

$$\begin{aligned} g(x) &= \psi(a) \text{ for } a \le x < x_1, \\ &= \psi(x_i + 0) \text{ for } x_i \le x < x_{i+1}\ (i =1, 2, \cdots, n-1). \\ &= 0 \text{ for } x = b \end{aligned}$$

Then g is decreasing, non–negative and bounded on $[a, b]$. Clearly the functions fg and $f\psi$ are summable on $[a, b]$. We have

$$\int_a^b fgd\mu - \int_a^b f\psi d\mu = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f\{g-\psi\}d\mu \quad \ldots \quad \ldots \quad \ldots \quad (7.12)$$

$$[x_0=a].$$

and

$$\left|\int_a^b gfdu - \int_a^b \psi fd\mu\right| \le \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (g-\psi)|f|d\mu$$

$$\le 2\ \psi\ (a).\ \varepsilon + \varepsilon\ \int_a^b |f|du = k \cdot \varepsilon \text{ (say)}$$

So

$$\int_a^b gfd\mu - k\varepsilon < \int_a^b \psi fd\mu < \int_a^b gfdu + k\varepsilon \quad \ldots \quad (7.13)$$

Let $h(x) = \int_a^x fd\mu$ for $a \le x \le b$. Then h is continuous on $[a, b]$. Denote by m and M the glb and lub of the function h on $[a, b]$.

Write

$$a_{i+1} = g(x_i+0) \text{ and } v_{i+1} = \int_{x_i}^{x_{i+1}} fd\mu\ (i = 0, 1, 2, \ldots, n-1).$$

Then

$$m \le v_1 + v_2 + \cdots + v_p \le M\ (p = 1,2, \cdots, n)$$

So by Abel's Lemma,

$$m\, a_1 \leq a_1 v_1 + a_1\, v_2 + \cdots + a_n v_n \leq M\, a_1$$

or

$$m\, \psi\,(a) \leq \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} g(x_i + 0)\, f d\mu \leq M\, \psi\,(a)$$

or

$$m\, \psi\,(a) \leq \int_a^b g f d\mu \leq M\, \psi\,(a) \qquad \cdots \qquad \cdots \qquad (7.14)$$

Combining (7.13) and (7.14) we get

$$m\, \psi\,(a) - k\, \varepsilon < \int_a^b \psi f d\mu < M\, \psi\,(a) + k\, \varepsilon.$$

Since ε is arbitrary we obtain

$$m\, \psi\,(a) \leq \int_a^b \psi f d\mu \leq M\, \psi\,(a) \qquad \cdots \qquad \cdots \qquad (7.15)$$

Write $\int_a^b \psi f d\mu = \lambda \psi\,(a)$.

Then using (7.15) we get $m \leq \lambda \leq M$.

So there is a point ξ in $[a,b]$ such that $\lambda = h\,(\xi) = \int_a^\xi f d\mu$.

Hence

$$\int_a^b \psi f d\mu = \psi(a) \int_a^\xi f d\mu$$

or

$$\int_a^b \{F(b) - F\} f d\mu = \{F(b) - F(a)\} \int_a^\xi f d\mu$$

or

$$\int_a^b F f d\mu = F(a) \int_a^\xi f d\mu + F(b) \int_\xi^b f d\mu.$$

Theorem 7.10. Let μ be a Lebesgue-stieltjes measure on the real line Ω and E be a measurable (μ) set with $\mu E < +\infty$. If the function f is summable on E, then for every $\varepsilon > 0$ there is a function g continuous on Ω such that

$$\int_E |f - g| d\mu < \varepsilon$$

Proof : We prove the theorem by the following steps. Choose any $\varepsilon > 0$.

(I) Suppose that f is bounded and non-negative on the set E. Determine a positive number k such that

$$0 \leq f(x) \leq k \text{ for all } x \in E.$$

Choose $\qquad \sigma = \dfrac{\varepsilon}{2(1 + \mu E)}$ and $\eta = \dfrac{\varepsilon}{4k}$.

Then by Theorem 4.11 there is a function g continuous on Ω such that

$$\mu E(\,|f - g| \geq \sigma) < \eta \text{ and } 0 \leq g(x) \leq k.$$

Write $E_1 = E(\,|f - g| < \sigma)$ and $E_2 = E(|f - g| \geq \sigma)$.

We have

$$\int_E |f-g|d\mu = \int_{E_1} |f-g|d\mu + \int_{E_2} |f-g|d\mu$$

$$\leq \sigma.\ \mu\ E_1 + 2k.\ \eta < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

(II) Suppose that f is non-negative and unbounded on E. Then there is a positive integer N such that

$$\int_E \{f - [f]_N\}d\mu < \frac{1}{2}\varepsilon.$$

By case I, there is a function g continuous on Ω such that

$$\int_E |[f]_N - g|d\mu < \frac{1}{2}\varepsilon.$$

We have

$$\int_E |f-g|d\mu \leq \int_E |f - [f]_N|d\mu + \int_E |[f]_N - g|d\mu$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

(III) Lastly suppose that f is any function summable on E.

By case II, there are functions g_1 and g_2 continuous on Ω such that

$$\int_E |f_+ - g_1|d\mu < \frac{1}{2}\varepsilon \text{ and } \int_E |f_- - g_2|d\mu < \frac{1}{2}\varepsilon.$$

Write $g = g_1 - g_2$. Then g is continuous on Ω. We have

$$f - g = (f_+ - g_1) - (f - g_2).$$

So

$$\int_E |f-g|d\mu \leq \int_E |f_+ - g_1|d\mu + \int_E |f_- - g_2|d\mu$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

Theorem 7.10A. Let μ be a Lebesgue-stieltjes measure on the real line Ω and let E be a measurable (μ) set with $\mu E < +\infty$. If the function f is square summable on E, then given any $\varepsilon > 0$ there is a function g continuous on Ω such that

$$\int_E |f-g|^2 du < \varepsilon.$$

Proof : We prove the theorem by the following steps. Choose any $\varepsilon > 0$.

(I) suppose that f is bounded on E. Then there is *a* positive number k such that

$$|f(x)| \leq k \text{ for all } x \in E.$$

Let $\sigma = \dfrac{\varepsilon}{2(1+\mu E)}$ and $\eta = \dfrac{\varepsilon}{8k^2}$.

By Theorem 4.11 thereis *a* function g continuous on Ω such that $\mu E\left(1f - g1 \geq \sqrt{\sigma}\right) < \eta$ and $|g(x)| \leq k$ for all $x \in \Omega$. Write

$E_1 = E\left(1|f - g| \geq \sqrt{\sigma}\right)$ and $E_2 = E\left(1f - g1 \geq \sqrt{\sigma}\right)$.

We have

$$\int_E |f - g|^2 d\mu = \int_{E_1} |f - g|^2 d\mu + \int_{E_2} |f - g|^2 d\mu$$

$$\leq \sigma\, \mu\, E_1 + 4k^2\, \mu\, E_2 < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

(II) Now suppose that f is unbounded on E. Since f is square summable on E, there is *a* positive number δ such that for any measurable set $e \subset E$ with $\mu\, e < \delta$,

$$\int_e f^2 d\mu < \frac{1}{8}\varepsilon.$$

Let $E_n = E\,(|\,f\,| \leq n)$ $(n = 1, 2, 3, \ldots.)$. Then the sets $E_1, E_2, E_3, \ldots$are measurable and $E_1 \subset E_2 \subset \ldots$ and $\lim \mu\, E_n = \mu\, E$. Choose *a* positive integer N such that for any $n \geq N$, $\mu E_n > \mu E - \delta$.

Write $A = E_N$ and $B = E/E_N$. Then $\mu\, B \leq \delta$. Since f is bounded on A, by case I, there is *a* function g continuous on Ω with $|\,g(x)\,| \leq N$ for $x \in \Omega$ such that

$$\int_A |f - g|^2 d\mu < \frac{1}{2}\varepsilon.$$

Clearly $|g(x)| \leq |f(x)|$ for all $x \in B$.

We have

$$\int_E |f - g|^2 d\mu = \int_A |f - g|^2 d\mu + \int_B |f - g|^2 d\mu < \frac{1}{2}\varepsilon + 4\int_B f^2 d\mu$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

Theorem 7.11. (Shwarz's Inequality).

Let the functions f and g be square summable on the set E. Then fg is summable on E and

$$\left|\int_E fg\,d\mu\right| \le \left(\int_E f^2 d\mu\right)^{\frac{1}{2}} \left(\int_E g^2 d\mu\right)^{\frac{1}{2}}.$$

Proof : For all $x \in E$ we have

$$|f(x)g(x)| \le \frac{1}{2}[f^2(x) + g^2(x)].$$

This gives that $|fg|$ is summable on E and so is fg.

Write $A^2 = \int_E f^2 d\mu, B = \int_E fg\,d\mu, C^2 = \int_E g^2 d\mu$.

If $A = 0$, then $f(x) = 0$ *a.e.* on E. This implies that $B = 0$. Therefore the result is true. We suppose that $A > 0$. Let λ be any real number. Then

$$\lambda^2 A^2 + 2\lambda B + C^2 = \int_E [\lambda f(x) + g(x)]^2 d\mu \ge 0.$$

Now $\lambda^2 A^2 + 2\lambda B + C^2 = \left(\lambda A + \frac{B}{A}\right)^2 + C^2 - \frac{B^2}{A^2}.$

Taking $\lambda = -\frac{B}{A}$ we see that

$$C^2 - \frac{B^2}{A^2} \ge 0 \text{ or } B^2 \le A^2C^2$$

So $|B| \le AC$,

that is, $$\left|\int_E f_g d\mu\right| \le \left(\int_E f^2 d\mu\right)^{\frac{1}{2}} \left(\int_E g^2 d\mu\right)^{\frac{1}{2}}.$$

Theorem 7.12. Let the functions f, g and h be square summable on the set E. Then

$$\left(\int_E |f - h|^2 d\mu\right)^{\frac{1}{2}} \le \left(\int_E |f - g|^2 d\mu\right)^{\frac{1}{2}} + \left(\int_E |g - h|^2 d\mu\right)^{\frac{1}{2}}.$$

Proof : We have for almost all x in E,

$$|f(x) - h(x)|^2 \leq |f(x) - g(x)|^2 + |g(x) - h(x)|^2$$
$$+ 2\,|f(x) - g(x)|\,|g(x) - h(x)|$$

So

$$\int \int_E |f - h|^2\, d\mu \leq \int_E |f - g|^2\, d\mu + \int_E |g - h|^2\, d\mu$$

$$+ 2 \int_E |f - g| \cdot |g - h|\, d\mu$$

$$\leq \int_E |f - g|^2\, d\mu + \int_E |g - h|^2\, d\mu$$

$$+ 2 \left(\int_E |f - g|^2\, d\mu\right)^{\frac{1}{2}} \left(\int_E |g - h|^2\, d\mu\right)^{\frac{1}{2}}$$

$$= \left\{\left(\int_E |f - g|^2\, d\mu\right)^{\frac{1}{2}} + \left(\int_E |g - h|^2\, d\mu\right)^{\frac{1}{2}}\right\}^2.$$

Therefore

$$\left(\int_E |f - h|^2\, d\mu\right)^{\frac{1}{2}} \leq \left(\int_E |f - g|^2\, d\mu\right)^{\frac{1}{2}} + \left(\int_E |g - h|^2\, d\mu\right)^{\frac{1}{2}}.$$

Theorem 7.13. Let the function f be square summable on $[a,b]$ and μ denote the usual Lebesgue measure on the real line. Then for any $\varepsilon > 0$ there is *a* polynomial h such that

$$\left(\int_a^b |f - h|^2\, d\mu\right)^{\frac{1}{2}} < \varepsilon.$$

Proof : Choose any $\varepsilon > 0$.

By Theorem 7.10A there is *a* function g continuous on $[a, b]$ such that

$$\left(\int_a^b |f - g|^2\, d\mu\right)^{\frac{1}{2}} < \frac{1}{2}\varepsilon.$$

By Weierstrass's Approximation Theorem there is a polynomial h such that

$$|g(x)-h(x)| < \sqrt{\frac{\varepsilon^2}{4(b-a)}} \quad \text{for all } x \in [a,b].$$

Then $$\int_a^b |g-h|^2\,d\mu \le \frac{\varepsilon^2}{4(b-a)}\int_a^b d\mu = \frac{1}{4}\varepsilon^2 .$$

By Theorem 7.12.

$$\left\{\int_a^b |f-h|^2\,d\mu\right\}^{\frac{1}{2}} \le \left(\int_a^b |f-g|^2\,d\mu\right)^{\frac{1}{2}} + \left(\int_a^b |g-h|^2\,d\mu\right)^{\frac{1}{2}}$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

CHAPTER—VIII

FUNCTIONS OF BOUNDED VARIATION, ABSOLUTELY CONTINUOUS FUNCTIONS AND THEIR DIFFERENTIABILITY

Functions of bounded variation.

Let f be a real-valued function on the closed interval $[a, b]$ and let $D : (a = x_0 < x_1 < x_2 < .... < x_n = b)$ be any subdivision of $[a, b]$. Write

$$V_D(f) = \sum_{\nu=1}^{n} |f(x_\nu) - f(x_{\nu-1})|.$$

Denote by $\mathcal{F}$ the set of all subdivisions D of $[a, b]$.

Let $V_a^b(f) = \sup \{V_D(f) : D \in \mathcal{F}\}$.

If $V_a^b(f)$ is finite we say that f is of bounded variation (or BV) on $[a,b]$.

Theorem 8.1. If f is BV on $[a,b]$, then f is BV on $[\alpha,\beta] \subset [a, b]$.

Proof : Let f be BV on $[a, b]$ and $[\alpha,\beta] \subset [a, b]$. Take any subdivision

$$D : (\alpha = x_0 < x_1 < x_2 < \cdots < x_n = \beta)$$

of $[\alpha,\beta]$. Clearly the points $a, x_0, x_1, x_2, ..., x_n, b$ form a subdivision $\tilde{D}$ of $[a, b]$.

We have

$$V_D(f) = \sum_{\nu=1}^{n} |f(x_\nu) - f(x_{\nu-1})|$$

$$\leq |f(\alpha) - f(a)| + \sum_{\nu=1}^{n} |f(x_\nu) - f(x_{\nu-1})| + |f(b) - f(\beta)|$$

$$= V_{\tilde{D}}(f) \leq V_a^b(f). \qquad \text{... (8.1)}$$

Since f is BV on $[a,b]$, $V_a^b(f)$ is finite. (8.1) gives that

$$V_\alpha^\beta(f) \leq V_a^b(f) < +\infty.$$

Hence f is BV on $[\alpha,\beta]$

Theorem 8.2. If f is BV on $[a,b]$ then f is bounded on $[a,b]$.

Proof : Suppose that f is BV on $[a,b]$. Take any point x in (a,b). The points a, x, b form a subdivision of $[a,b]$. So

$$|f(x)-f(a)| + |f(b)-f(x)| \le V_a^b(f).$$

We have

$$|f(x)| \le |f(a)| + |f(x)-f(a)| \le |f(a)| + V_a^b(f).$$

Let $k = max \{ |f(b)|, |f(a)| + V_a^b(f)\}$.

Then clearly

$$|f(x)| \le k \text{ for all } x \in [a,b].$$

Hence f is bounded on $[a,b]$.

Theorem 8.3. Let the functions f and g be BV on $[a, b]$. Then $f \pm g$ and fg are BV on $[a,b]$.

Proof : Let $\phi = f \pm g$ and $\psi = fg$, Take any subdivision

$D : (a = x_0 < x_1 < x_2 < \cdots < x_n = b.)$

of $[a, b]$. We have

$$V_D(f) = \sum_{\nu=1}^{n} |\phi(x_\nu) - \phi[x_{\nu-1})|$$

$$\le \sum_{\nu=1}^{n} |f(x_\nu) - f(x_{\nu-1})| + \sum_{\nu=1}^{n} |g(x_\nu) - g(x_{x=1})|$$

$$\le V_a^b(f) + V_a^b(g)$$

So $$V_a^b(\phi) \le V_a^b(f) + V_a^b(g) < +\infty.$$

Hence ϕ is BV on $[a,b]$.

Again, we have

$$|\psi(x_\nu) - \psi(x_{\nu-1})| = |f(x_\nu)g(x_\nu) - f(x_{\nu-1})\, g(x_{\nu-1})|$$
$$\le |f(x_\nu) - f(x_{\nu-1})|\, |g(x_\nu)| + |f(x_{\nu-1})|\, |g(x_\nu) - g(x_{\nu-1})|$$
$$\le k\{ |f(x_\nu) - f(x_{\nu-1})| + |g(x_\nu) - g(x_{\nu-1})|\},$$

where $$k = \sup \{ |f(x)|, |g(x)| : a \le x \le b\}.$$

So, $$V_D(\psi) = \sum_{\nu=1}^{n} |\psi(x_\nu) - \psi(x_{\nu-1})|$$

$$\le k\{V_D(f) + V_D(g)\}$$

$$\le k \left\{V_a^b(f) + V_a^b(g)\right\}.$$

This gives that

$$V_a^b(\psi) \le k\left\{V_a^b(f) + V_a^b(g)\right\} < +\infty.$$

Hence ψ is BV on $[a,b]$.

Theorem 8.4. Let f be BV on $[a, b]$ and let $|f(x)| \geq \sigma > 0$ for all $x \in [a,b]$. Then $1/f$ is BV on $[a,b]$.

The proof is left as an exercise

Theorem 8.5. Let the function f possess finite derivative at each point of the interval $[a,b]$. If f' is bounded on $[a,b]$, then f is BV on $[a, b]$.

Proof : Suppose that f' is bounded on $[a, b]$. Then there is a positive number k such that

$$|f'(x)| \leq k \text{ for all } x \in [a, b].$$

Let $D : (a = x_0 < x_1 < x_2 < \cdots < x_n = b)$ be any subdivision of $[a,b]$. We have by mean-Value theorem

$$f(x_\nu) - f(x_{\nu-1}) = (x_\nu - x_{\nu-1}) f'(\xi),$$

where $\quad x_{\nu-1} < \xi_\nu < x_\nu \ (\nu = 1, 2, \ldots, n).$

So
$$V_D(f) = \sum_{\nu=1}^{n} |f(x_\nu) - f(x_{\nu-1})|$$

$$\leq k \sum_{\nu=1}^{n} (x_\nu - x_{\nu-1}) = k(b - a).$$

This gives that

$$V_a^b(f) \leq k(b-a).$$

Hence f is BV on $[a,b]$.

Theorem 8.6. If f is monotone on $[a,b]$, then f is BV on $[a,b]$.

Proof : Suppose that f is increasing on $[a,b]$. Take any subdivision

$$D = (a_0 < x_1 < x_2 < \cdots < x_n = b)$$

of $[a, b]$. Then

$$V_D(f) = \sum_{\nu=1}^{n} |f(x_\nu) - f(x_{\nu-1})|$$

$$= \sum_{\nu=1}^{n} \{f(x_\nu) - f(x_{x-1})\} = f(b) - f(a).$$

This gives that

$$V_a^b(f) = f(b) - f(a).$$

Hence f is BV on $[a, b]$.

Theorem 8.7. Let f be any real-valued function on $[a,b]$ and let $a < c < b$.

Then
$$V_a^b(f) = V_a^c(f) + V_c^b(f).$$

Proof : We first suppose that $V_a^c(f)$ and $V_c^b(f)$ are finite. Choose any $\varepsilon > 0$.

Then there is a subdivision $a = x_0 < x_1 < x_2 < \cdots < x_n = c$ of $[a,c]$ and a subdivision $c = y_0 < y_1 < y_2 < \cdots < y_m = b$ of $[c, b]$ such that

$$\left.\begin{array}{l} \sum\limits_{\nu=1}^{n} |f(x_\nu) - f(x_{x-1})| > V_a^c(f) - \frac{1}{2}\varepsilon \\ \sum\limits_{\nu=1}^{m} |f(y_\nu) - f(y_{\nu-i})| > V_c^b(f) - \frac{1}{2}\varepsilon \end{array}\right\} \quad \cdots \quad (8.2)$$

Clearly the points $x_0, x_1, x_2, \cdots, x_n = y_0, y_1, y_2, \ldots, y_m$ form a subdiviaion of $[a, b]$. We have

$$\sum_{\nu=1}^{n} |f(x_\nu) - f(x_{\nu-1})| + \sum_{\nu=1}^{m} |f(y_\nu) - f(y_{\nu-1})| \le V_a^b(f) \quad \ldots. \quad (8.3)$$

From (8.2) and (8.3) we get

$$V_a^c(f) + V_c^b(f) - \varepsilon < V_a^b(f).$$

Since $\varepsilon > 0$ is arbitrary we obtain

$$V_a^c(f) + V_c^b(f) \le V_a^b(f) \quad \cdots \quad \cdots \quad (8.4)$$

Now let $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be any subdivision of $[a, b]$. There is a positive integer m such that $x_{m-1} < c \le x_m$. The points $x_0, x_1, x_2, \cdots, x_{m-1}, c$ form a subdision of $[a,c]$ and the points $c, x_m, x_{m+1}, \cdots, x_n$ form a subdivision of $[c,b]$. So we have

$$\sum_{\nu=1}^{n} |f(x_\nu) - f(x_{\nu-1})|$$

$$= \sum_{\nu=1}^{m-1} |f(x_\nu) - f(x_{\nu-1})| + |f(x_m) - f(x_{m-1})| + \sum_{\nu=m+1}^{n} |f(x_\nu) - f(x_{\nu-1})|$$

$$\le \left\{ \sum_{\nu=1}^{m-1} |f(x_\nu) - f(x_{\nu-1})| + |f(c) - f(x_{m-1})| \right\}$$

$$+ \left\{ |f(x_m) - f(c)| + \sum_{\nu=m+1}^{n} |f(x_\nu) - f(x_{\nu-1})| \right\}$$

$$\le V_a^c(f) + V_c^b(f).$$

This gives that

$$V_a^b(f) \leq V_a^c(f) + V_c^b(f). \qquad \cdots \qquad (8.5)$$

From (8.4) amd (8.5) we obtain

$$V_a^b(f) = V_a^c(f) + V_c^b(f). \qquad \cdots \qquad (8.6)$$

Next, suppose that one of $V_a^c(f)$ *and* $V_c^b(f)$ *say* $V_a^c(f)$ is infinite. Choose any positive number G. Then there is a subdivision

$a = x_0 < x_1 < x_2 < \cdots < x_n = c$ of $[a, c]$ such that

$$\sum_{\nu=1}^{n} |f(x_\nu) - f(x_{\nu-1})| > G.$$

The points $x_0, x_1, x_2, \ldots, x_n, b$ form a subdivision of $[a, b]$. So we have

$$V_a^b(f) \geq \sum_{\nu=1}^{n} |f(x_\nu) - f(x_{\nu-1})| + |f(b) - f(x_n)| > G.$$

This gives that $V_a^b(f) = +\infty$. Hence the relation (8.6) holds.

Theorem 8.8. Let the function f be BV on $[a, b]$. Then f can be expresed in the from $f = g - h$, where g and h are increasing on $[a, b]$.

Proof : Define the functions g and h on $[a, b]$ as follows.

$g(a) = 0$ and $g(x) = V_a^x(f)$ for $a < x \leq b$,

and $\quad h(x) = g(x) - f(x)$ for $a \leq x \leq b$.

Let x_1 and x_2 $(x_1 < x_2)$ be any two points in $[a, b]$. We have

$$g(x_2) = V_a^{x_2}(f) = V_a^{x_1}(f) + V_{x_1}^{x_2}(f)$$

$$= g(x_1) + V_{x_1}^{x_2}(f) \geq g(x_1) \qquad \cdots \quad \cdots \qquad (8.7)$$

$$h(x_2) - h(x_1) = \{g(x_2) - f(x_2)\} - \{g(x_1) - f(x_1)\}$$
$$= \{g(x_2) - g(x_1)\} - \{f(x_2) - f(x_1)\}$$

$$\geq V_{x_1}^{x_2}(f) - |f(x_2) - f(x_1)| \qquad \cdots\cdots \quad \cdots\cdots \qquad (8.8)$$

Take any point $\bar{x}$ with $x < \bar{x} < x_2$. The points $x_1, \bar{x}, x_2$ form a subdivision of $[x_1, x_2]$. So

$$|f(\bar{x}) - f(x_1)| + |f(x_2) - f(\bar{x})| \leq V_{x_1}^{x_2}(f)$$

Since $$| f(x_2) - f(x_1) | \leq | f(\bar{x}) - f(x_1)| + | f(x_2) - f(\bar{x})|,$$
we get
$$| f(x_2) - f(x_1)| \leq V_{x_1}^{x_2}(f) \qquad \dots \qquad \dots \qquad (8.9)$$

From (8.8) amd (8.9) we have
$$h(x_2) - h(x_1) \geq 0. \qquad \dots \qquad \dots \qquad (8.10)$$

Since (8.7) and (8.10) hold for any two points x_1, x_2 $(x_1 < x_2)$ in $[a, b]$, it follows that g and h are incrcasing on $[a, b]$. From definition of h we get
$$f(x) = g(x) - h(x) \quad \text{for all } x \in [a, b].$$

Theorem 8.9. Let the function f be BV on the interval $[a, b]$. Then f is continuous on $[a,b]$ except possibly a countable set.

Proof : By Theorem 8·8 we have $f = g - h$ where g and h are increasing on $[a, b]$. Let E_1 and E_2 denote the sets of the points of discontinmty of g and h respectively in $[a, b]$. Then by Theorem 1.24. the sets E_1 and E_2 are countable and so $E_1 \cup E_2$ is countable. Let $x \in [a,b] \setminus (E_1 \cup E_2)$. Then g and h are continuous at x and so f is continuous at x. This proves the theorem.

Theorem 8.10. Let the function f be BV on $[a, b]$ and let $g(x) = V_a^x(f)$ for $a < x \leq b$, $g(a) = 0$. Then g is continuous at each point of continuity of f in $[a, b]$.

Proof : Clearly g is increasing on $[a,b]$. Suppose that f is continuous at c, where $a < c < b$. Choose any $\varepsilon > 0$. There is a subdivision $a = x_0 < x_1 < x_2 < \cdots < x_n = c$ of $[a, c]$ and a subdivision $c = y_0 < y_1 < y_2 < \cdots < y_m = b$ of $[c,b]$ such that

$$\sum_{\nu=1}^{n} | f(x_\nu) - f(x_{\nu-1})| > V_a^c(f) - \frac{1}{2}\varepsilon = g(c) - \frac{1}{2}\varepsilon \qquad \dots \qquad \dots \qquad (8.11)$$

$$\sum_{\nu=1}^{m} | f(y_\nu) - f(y_{\nu-1})| > V_c^b(f) - \frac{1}{2}\varepsilon = g(b) - g(c)\frac{1}{2}\varepsilon \qquad \dots \qquad \dots \qquad (8.12)$$

We choose a positive number δ with
$$0 < \delta < \min\{c - x_{n-1}, y_1 - c\}. \text{ such that}$$
$$| f(x) - f(c)| < \frac{1}{2}\varepsilon \text{ for all } x \in (c - \delta, c + \delta).$$

Take any point x in $(c - \delta, c)$. We have by (8.11)

$$g(c) - \frac{1}{2}\varepsilon < \left\{ \sum_{\nu=1}^{n-1} | f(x_\nu) - f(x_{\nu-1})| + | f(x) - f(x_{n-1})| \right\}$$

$$+|f(c)-f(x)|<V_a^x(f)+\frac{1}{2}\varepsilon=g(x)+\frac{1}{2}\varepsilon.$$

or $$g(c)-\varepsilon<g(x)\leq g(c)<g(c)+\varepsilon \quad \ldots \quad (8.13)$$

Now take any point x in $(c, c+\delta)$. We get from (8.12)

$$g(b)-g(c)-\frac{1}{2}\varepsilon<\left\{\sum_{\nu=2}^{m}|f(y_\nu)-f(y_{\nu-1})|+|f(y_1)-f(x)|\right\}$$

$$+|f(x)-f(c)|$$

$$<V_x^b(f)+\frac{1}{2}\varepsilon=g(b)-g(x)+\frac{1}{2}\varepsilon$$

or $$g(c)\leq g(x)<g(c)+\varepsilon$$

or $$g(c)-\varepsilon<g(x)<g(c)+\varepsilon \quad \ldots \quad (8.14)$$

From (8.13) and (8.14) we obtain

$$|g(x)-g(c)|<\varepsilon \text{ for all } x \text{ in } (c-\delta, c+\delta).$$

Hence g is continuous at c.

If $c = a$ or b as above can show that g is continuous at c.

Lipschitz condition. Let f be a real-valued function on $[a, b]$. f is said to satisfy Lipschitz condition on $[a, b]$ if there is a positive number k such that for all x', x'' in $[a, b]$,

$$|f(x')-f(x'')|\leq k\,|x'-x''|.$$

Examples 8.1. (1) Let $f(x) = x \sin\frac{\pi}{x}$ for $0<x\leq 1$ and $f(0) = 0$. Show that f is not BV on $[0,1]$.

Solution. Consider the subdivision

$$D:(0<x_{2n-1}<\cdots<x_3<x_2<x_2<x_1=1),$$

where $x_\nu=\dfrac{2}{\nu+1}$ $(\nu = 1, 2, 3, \ldots, 2n)$.

We have

$V_D(f) = |f(x_1)-f(x_2)| + |f(x_2)-f(x_3) + |f(x_3)-f(x_1)| + |f(x_4) - f(x_5)| + \cdots + |f(x_{2x-1})-f(x_{2n})| + |f(x_{2n})-f(0)|$

Now $f(x_{2\nu-1})=\dfrac{2}{2\nu}\sin(\pi\nu)=0$

and $f(x_{2\nu})=\dfrac{2}{2\nu+1}\sin\left(\pi\nu+\dfrac{\pi}{2}\right)=\dfrac{2(-1)^\nu}{2\nu+1}.$

So $V_D(f) = 2\left(\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots+\frac{1}{2^{n+1}}\right) \to \infty$ as $n \to \infty$.

Therefore f is not of bounded variation on $[0,1]$.

(2) Let f be *a* real valued function on $[a, b]$ and let it satisfy lipschitz's condition on $[a,b]$. Then f is BV on $[a, b]$.

Solution. Since f satisfies lipslitz's condition on $[a, b]$, there is *a* positive number k such that

$$| f(x') - f(x'')| \leq k \mid x' - x''| \qquad \ldots \qquad \ldots \qquad (8.15)$$

for all x', x'' in $[a, b]$

Take any subdivision

D : $(a = x_0 < x_1 < x_2 < \cdots < x_n = b)$ of $[a, b]$.

We have

$$V_D(f) = \sum_{\nu=1}^{n} |f(x_\nu) - f(x_{\nu-1})|$$

$$\leq k \sum_{\nu=1}^{n} (x_\nu - x_{\nu-1}) \qquad \text{[By (8.16)]}$$

$$\leq k(b-a).$$

This gives that

$$V_a^b(f) \leq k\,(b-a).$$

Hence f is BV on $[a,b]$.

(3) Let $f(x) = \dfrac{1}{q^3}$ when $x = \dfrac{p}{q}$, where pand q are postive integers and $(p,q) = 1$and $f(x) = 0$ *when* x is irrational; and $f(0) = f(1) = 1$.

Show that f is BV on $[0, 1]$

Absolutely continuous functions

Let f be *a* realvalued function on the interval $[a, b]$. The function f is said to be absolutely continuous on $[a, b]$ if for every $\varepsilon > 0$ there is *a* $\delta > 0$ such that for every set of non-overlapping intervals $\{(a_i, b_i)\}$ in $[a,b]$

$$\sum_i |f(b_i) - f(a_i)| < \varepsilon \text{ whenever} \sum_i (b_i - a_i) < \delta.$$

It is easy to see that absolute continuity implies continuity and uniform continuity.

Theorem 8.11. let f be a real-valued function on the interval $[a,b]$. If f satisfies Lipschitz's condition on $[a, b]$, then f is absolutely continuous on $[a,b]$.

Proof : Since f satisfies Lipschitz's condition on $[a,b]$, there is a positive number k such that for all x', x'' in $[a,b]$,

$$|f(x') - f(x'')| \leq k\,|x'-x''|. \quad \ldots \quad (8.17)$$

Choose any $\varepsilon > 0$. Take $\delta = \dfrac{\varepsilon}{k}$.

Let $\{(a_i, b_i)\}$ be any set of non-overlapping intervals in $[a,b]$ with $\Sigma(b_i - a_i) < \delta$.

Then

$$\sum_i |f(b_i) - f(a_i)| \leq k \sum_i (b_i - a_i) < k\frac{\varepsilon}{k} = \varepsilon.$$

Hence f is absolutely continuous on $[a, b]$.

Corollary 8.11.1. Let f be a real-valued function on the interval $[a, b]$. If f possesses bounded derivative on $[a, b]$, then f is absolutely continuous on $[a, b]$.

Proof : Let x_1 and x_2 $(x_1 < x_2)$ be any two points in $[a, b]$. By Mean Value Theorem

$$f(x_2 - f(x_1) = (x_2 - x_1)\, f'(\zeta) \;(x_1 < \xi < x_2).$$

So $\qquad |f(x_2) - f(x_1)| \leq k(x_2 - x_1),$

where $\qquad k = \sup\{|f'(x)| : a \leq x \leq b\}.$

Thus f satisfies Lipshitz's condition on $[a, b]$. Hence f is AC on $[a, b]$.

Theorem 8.12. Let the functions f and g be absolutely continuous on $[a, b]$. Then $f \pm g$ and fg are absolutely continuous on $[a, b]$.

Proof : Since f and g are absolutely continuous on $[a, b]$, they are bounded there. So there is a positive number A such that

$$|f(x)| \leq A \text{ and } |g(x)| \leq A \text{ for all } x \in [a, b].$$

(i) Choose any $\varepsilon > 0$. There is a $\delta > 0$ such that for any set $\{(a_i, b_i)\}$ of non-overlapping intervals in $[a, b]$

$$\left.\begin{aligned} &\sum_i |f(b_i) - f(a_i)| < \frac{1}{2}\varepsilon \\ \text{and } &\sum_i |g(b_i) - g(a_i)| < \frac{1}{2}\varepsilon \end{aligned}\right\} \text{whenever } \sum_i (b_i - a_i) < \delta.$$

Write $\phi = f \pm g$. Then

$$\sum_i |\phi(b_i) - \phi(a_i)| \le \sum_i |(b_i) - f(a_i)| + \sum_i |g(b_i) - g(a)|$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

Hence ϕ is absolutely continuous on $[a, b]$.

(ii) Write $\phi = fg$. Choose any $\varepsilon > 0$.

There is a $\delta > 0$ such that for any set $\{a_i, b_i)\}$ of non-overlapping intervals in $[a,b]$

$$\sum_i |f(b_i) - f(a_i)| < \frac{\varepsilon}{2A} \text{ and } \sum_i |g(b_i)| - g(a_i)| < \frac{\varepsilon}{2A}.$$

whenever $\sum_i (b_i - a_i) < \delta$.

We have

$$\begin{aligned} |\phi(b_i) - \phi(a_i)| &= |f(b_i)g(b_i) - f(a_i)g(a_i)| \\ &\le |f(b_i) - f(a_i)|\,|g(b_i)| + |f(a_i)|\,|g(a_i) - g(a_i)| \\ &\le A\{|f(b_i) - f(a_i)| + |g(b_i) - g(a_i)|\} \\ &< A\left(\frac{\varepsilon}{2A} + \frac{\varepsilon}{1A}\right) = \varepsilon. \end{aligned}$$

Hence ϕ is absolutely continuous on $[a, b]$.

Theorem 8.13. Let f be absolutely continuous on $[a, b]$ and $f(a) \ge \sigma > 0$ for all x in $[a,b]$. then $\frac{1}{f}$ is absolutely continuous on $[a,b]$.

The proof is left as an exercise.

Theorem 8.14. Let the function f be absolutely continuous on $[a, b]$. Then f is BV on $[a, b]$.

Proof : We can determine a $\delta > 0$ such that for any set $\{(a_i, b_i)\}$ of non-overlapping intervals in $[a, b]$ with $\sum_i (b_i - a_i) < \delta$,

$$\sum_i |f(b_i) - f(a_i)| < 1 \qquad \dots \qquad \dots \qquad (8.17)$$

Let $a = c_0 < c_1 < c_2 < \cdots < c_N = b$ be any subdivision of $[a, b]$ with $(c_r - c_{r-1}) < \delta$ $(r = 1, 2, \dots, N)$.

Consider the interval $[c_{r-1}, c_r]$.

Let $c_{r-1} = x_0 < x_1 < x_2 < \cdots < x_n = c_r$ be any subdivision of $[c_{r-1}, c_r]$.

Clearly $\{(x_{i-1}, x_i)\}$ is a set of non-overlapping intervals in $[c_{r-1}, c_r]$ with $\sum_i (x_i - x_{i-1}) < \delta$.

So by (8·17) we have

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| < 1$$

This implies that

$$V_{c_{r-1}}^{c_r}(f) \leq 1.$$

So f is BV on $[c_{r-1}, c_r]$ and hence f is BV on $[a, b]$.

Theorem 8.15. Let the function f be summable (μ) on $[a,b]$, where μ is the usual lebesgue measure on the real line. Define ϕ on $[a, b]$ as follows.

$$\phi(x) = \int_a^x f(t)dt \text{ for } a \leq x \leq b.$$

Then ϕ is absolutely continuous on $[a, b]$.

Proof : Choose any $\varepsilon > 0$. Then there is a $\delta > 0$ such that for any measurable (μ) subset e of $[a, b]$ with $\mu e < \delta$

$$\int_e |f| d\mu < \varepsilon \qquad \ldots \quad \ldots \quad \ldots \qquad (8.18)$$

Take any set $\{(a_i, b_i)\}$ of non-overlapping intervals in $[a, b]$ with $\sum_i (b_i - a_i) < \delta$.

Let $e = \cup_i (a_i, b_i)$ Then e is a measurable set with $\mu e = \sum_i (b_i - a_i) < \delta$.

So by (8·18)

$$\int_e |f| d\mu < \varepsilon \qquad \ldots \quad \ldots \quad \ldots \qquad (8.19)$$

Now $\qquad \sum_i |\phi(b_i)| - \phi(a_i)|$

$$= \sum_i \left| \int_{a_i}^{b_i} f(t)dt \right|$$

$$\leq \sum_i \int_{a_i}^{b_i} |f| d\mu = \int_e |f| d\mu < \varepsilon. \text{ [by (8.19)].}$$

Hence ϕ is absolutely continuous on $[a, b]$.

Differentiability of BV and AC functions

Derived numbers.

Let $f: I \to \Omega$, where I is an interval which is not void or *a* singleton set. The number λ (finite or infinite) is said to be *a* derived number of f at the point $x_0 \in I$ if there is *a* null sequence $\{h_n\}$ $(h_n \neq 0)$ with $x_0 + h_n \in I$ such that

$$\lambda = \lim_{n\to\infty} \frac{f(x_0 + h_n) - f(x_0)}{h_n}.$$

A derived number of f at $x_0 \in I$ is denoted by $Df(x_0)$.

Note 8.1. *A* sequence $\{h_n\}$ of real numbers is said to be *a* null sequence if it converges to 0 (zero).

Theorem 8.16. Let $f: I \to \Omega$, where I is an interval which is not void or *a* singleton set. Then derived numbers of f exist at each point of I.

Proof : Let $x_0 \in I$. Take any null sequence $\{h_n\}$ $(h_n \neq 0)$ with $x_0 + h_n \in I$. Write

$$a_n = \frac{f(x_0 + h_n) - f(x_0)}{h_n} \quad (n = 1, 2, 3, \dots).$$

Let $\lambda = \lim_{n\to\infty} \inf a_n$. λ may be finite or infinite. In any case we can choose *a* sequence $\{n_k\}$ $(n_1 < n_2 < n_3 < \cdots)$ of positive integers such that

$$\lambda = \lim_{k\to\infty} a_{n_k}.$$

Write $p_k = h_{n_k}$ $(k = 1, 2, 3, \dots)$. Then $\{p_k\}(p_k \neq 0,\ x_0 + p_k \in I)$ is *a* null sequence and

$$\lambda = \lim_{k\to\infty} \frac{f(x_0 + p_k) - f(x_0)}{p_k}.$$

Thus λ is *a* derived number of f at x_0.

This proves the theorem.

Theorem 8.17. Let $f: I \to \Omega$, where I is an interval which is not void or *a* singleton set and $x_0 \in I$. Then $f'(x_0)$ exists if and only if all the derived numbers of at x_0 are equal.

Proof : First suppose that $f'(x_0)$ exists.

Then

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} \quad (h \neq 0,\ x_0 + h \in I).$$

Let $\{h_n\}$ $(h_n \neq 0,\ x_0 + h_n \in I)$ be any null sequence. Then clearly

$$\lim_{n \to \infty} \frac{f(x_0+h_n)-f(x_0)}{h_n} = f'(x_0).$$

This gives that all the derived numbers of f at x_0 are equal to $f'(x_0)$.

Next, suppose that all the derived numbers of f at $x_0 \in I$ are equal to l(say). Let $\{h_n\}$ $(h_n \neq 0,\ x_0 + h_n \in I)$ be any null sequence.

$$\text{Let } \lambda = \lim_{n \to \infty} \inf a_n \text{ and}$$

$$\Lambda = \lim_{n \to \infty} \sup a_n.$$

where

$$a_n = \frac{f(x_0+h_n)-f(x_0)}{h_n} \quad (n = 1, 2, 3, \ldots).$$

We can choose two subsequences $\{n_k\}$ $(n_1 < n_2 < n_3 < \cdots)$ and $\{m_k\}$ $(m_1 < m_2 < m_3 < \cdots)$ such that

$$\lambda = \lim_{k \to \infty} a_{n_k} \text{ and } \Lambda = \lim_{k \to \infty} a_{m_k}.$$

Charly λ and Λ are derived numbers of f at x_0. Therefore $\lambda = l = \Lambda$. This gives that $a_n \to l$ as $n \to \infty$ which implies that

$$\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} \ (h \neq 0, x_0 + h \in I) \text{ exists and equals}$$

l, that is $f'(x_0)$ exists and $f'(x_0) = l$.

Vilali covering.

Let μ denote the usual Lebesgue measure on the real line Ω and let $E \subset \Omega$; and $\mathcal{F}$ be *a* family of closed intervals none of which is void or *a* singleton set. If for every $x \in E$, there is *a* sequence $\{d_i\}$ in $\mathcal{F}$ such that $x \in d_i$ for all i and $\mu\, d_i \to 0$ as $i \to \infty$, then we say that the family $\mathcal{F}$ covers the set E in the sense of Vitali.

Theorem 8.18. (G. Vitali). Let E be *a* bounded sub of Ω and let the family $\mathcal{F}$ of closed intervals covers the set E in the sense of Vitali. Then for any $\varepsilon > 0$ we can select finite number of pairwise disjoint closed intervals $d_1, d_2, \ldots, d_n$ from the family $\mathcal{F}$ such that

$$\sum_{i=1}^{n}\mu^*(E \cap d_i) > \mu^*E - \varepsilon \text{ and } \sum_{i=1}^{n}\mu(d_i) < \mu^* E + \varepsilon.$$

Proof : Let $\varepsilon > 0$ be given. There is *a* bounded open set $G \supset E$ with $\mu G < \mu^*(E) + \varepsilon$.

Let $\mathscr{F}_0 = \{v : v \in \mathscr{F} \text{ and } v \subset G\}$.

Clearly $\mathscr{F}_0$ is also *a* Vitali covering of the set E. Choose *a* closed interval $d_1 \in \mathscr{F}_0$. Let

$l_1 = \sup\,\{\mu(v) : v \in \mathscr{F}_0 \text{ and } v \subset G\backslash d_1\}$.

Now choose d_2 in $\mathscr{F}_0$ with

$$d_2 \subset G \mid d_1 \text{ and } \mu d_2 > \frac{1}{2}l_1.$$

Let

$$l_2 = \text{Sup }\{\mu(v) : v \in \mathscr{F}_0 \text{ and } G\backslash\, (d_1 \cup d_2)\}.$$

Next, Choose d_3 in $\mathscr{F}_0$ with

$$d_3 \subset G\backslash\{d_1 \cup d_2\} \text{ and } \mu d_3 > \frac{1}{2}l_2.$$

Continuing the above process we obtain *a* sequence $\{d_k\}$ of closed intervals in $\mathscr{F}_0$ satisfying the following conditions.

(1) $d_n \subset G|(d_1 \cup d_2 \cup \ldots \cup d_{n-1})$

(2) $\mu d_n > \frac{1}{2}l_{n-1}$, where

$l_n = \sup\,\{\mu(v) : v \in \mathscr{F}_0 \text{ and } v \subset G\backslash(d_1 \cup d_2 \cup \cdots \cup d_{n-1})\}$.

Clearly the intervals $d_1, d_2, d_3, \ldots\ldots$are pairwise disjoint and $d_k \subset G$ for all k.

Write $A_n = \bigcup_{i=1}^{n} d_i$ and $\bigcup_{i=1}^{\infty} d_i = A$.

We have $A \subset G$ and so

$$\sum_{i=1}^{\infty} \mu(d_i) = \mu(A) \leq \mu\,(G) < \mu^*(E) + \varepsilon \qquad \ldots \quad (8.20)$$

We now show that $\mu^*\,(E \backslash A) = 0$.

Choose any $\eta > 0$. From (8.20) it follows that the series $\Sigma\mu(d_i)$ is convergent. So there is *a* positive integer N such that

$$\sum_{i=N+1}^{\infty} \mu(d_i) < \frac{1}{5}\eta.$$

Write $B = E\backslash A_N$. Take any $x \in B$.

There is a sequence $\{I_i\}$ of closed intervals in $\mathcal{F}_0$ such that $x \in I_i$ for all i and $\mu(I_i) \to 0$ as $i \to \infty$. Since $x \in G\backslash A_N$ and $G\backslash A_N$ is open, there is a $\delta > 0$ such that $(x - \delta, x + \delta) \subset G\backslash A_N$. Choose positive integer k such that

$\mu(I_i) < \frac{1}{2}\delta$ when $i \geq k$. Take $d = I_k$. Then $d \subset (x - \delta, x + \delta) \subset G\backslash A_N$.

If possible let $d \subset G\backslash A_n$ for all n.

Then $\mu(d) \leq l_n$ for all n. Since the series $\Sigma\mu(d_n)$ is convergent, $\mu(d_n) \to 0$ as $n \to \infty$. Again, since $l_n < 2\mu(d_{n+1})$, $l_n \to 0$ as $n \to \infty$. This contradicts the fact that $l_n \geq \mu d > 0$ for all n. So there is a positive integer m such that $d \subset G\backslash A_v$ for $v = 1, 2, \ldots, m-1$ but $d \not\subset G\backslash A_m$. This gives that $m > N$ and $d \cap d_m \neq \phi$. Since $d \cap A_{m-1} = \phi$ we have $d \subset G\backslash d_{m-1}$ which gives that $\mu d \leq l_{m-1} < 2\mu(d_m)$.

Let D_i denote the closed interval concentric with d_i and $\mu(D_i) = 5\mu(d_i)$. Let x_m be the mid-point of d_m and $z \in \alpha \cap d_m$. Then $|x - x_m| \leq |x - z| + |z - x_m| \leq \mu(d) + \mu\frac{1}{2}(d_m) < 2\mu(d_m) + \frac{1}{2}\mu(d_m) = \frac{5}{2}\mu d_m)$. So $x \in D_m$. Therefore

$B \subset \bigcup_{i=N+1}^{\infty} D_i$ and

$$\mu(B) \leq \sum_{i=N+1}^{\infty} \mu(D_i) = 5\sum_{i=N+1}^{\infty} \mu(d_i) < \eta.$$

Since $E\backslash A \subset B$ we have $\mu^*(E\backslash A) < \eta$. So $\mu^*(E\backslash A) = 0$.

Now $\quad E = (E \cap A) \cup (E\backslash A)$.

So $$\mu^*(E) = \mu^*(E \cap A) = \sum_{i=1}^{\infty}\mu^*(E \cap d_i).$$

Hence there is a positive integer n such that

$$\sum_{i=1}^{n}\mu^*(E \cap d_i) > \mu^*(E) - \varepsilon.$$

From (8.20) we get

$$\sum_{i=1}^{n}\mu(d_i) < \mu^*(E) + \varepsilon.$$

Lemma 8.1. Let $f : [a, b] \to \Omega$ be BV on $[a, b]$ and let E denote the set of all points in $[a, b]$ where at least one derived number of f is not finite. Then $\mu^*(E) = 0$.

Proof : If possible, let $\mu^*(E) > 0$. Choose any ε with $0 < \varepsilon < \frac{1}{2}\mu^*(E)$. Let

A be any positive number. Take any $x \in E$. Then there is a null sequence $\{h_n\}$ $(h_n \neq 0, x + h_n \in [a, b])$ such that for all n

$$\frac{|f(x+h_n)-f(x)|}{|h_n|} > A$$

or

$$|f(x+h_n) - f(x)| > A|h_n|. \quad \ldots \quad \ldots \quad (8.21)$$

Write $d(x,h_n) = [x, x + h_n]$ {or $[x+h_n, x]$}. Thus to each $x \in E$ we get a sequence of closed intervals $\{d(x,h_n)\}$ such that $\mu d(x, h_n) \to 0$ as $n \to \infty$. Let $\mathscr{F}$ denote the family of all closed intervals thus associated with the points of the set E. Then clearly $\mathscr{F}$ covers the set E in the sense of Vitali. Hence there exist finite number of pairwise disjoint closed intervals $\Delta_1, \Delta_2, \ldots, \Delta_p$ in family $\mathscr{F}$ such that

$$\sum_{i=1}^{p} \mu(\Delta_i) \geq \sum_{i=1}^{p} \mu^*(\Delta_i \cap E) > \mu^*(E) - \varepsilon \quad \ldots \quad \ldots \quad (8.22)$$

Let $\Delta_i = [a_i, b_i]$. Then from (8.21) and (8.22) we get

$$\sum_{i=1}^{p} |f(b_i) - f(a_i)| > A \sum_{i=1}^{p} \mu(\Delta_i) > A\{\mu^*(E) - \varepsilon\}.$$

This gives that

$$V_a^b(f) > \frac{1}{2} A\mu^*(E).$$

Since A is arbitrary, it follows that

$$V_a^b(f) = +\infty.$$

This contradicts our hypothisis.

Hence $\mu^*(E) = 0$.

Lemma 8.2. Let $f : [a, b] \to \Omega$ be increasing and let E denote the set of all points in $[a, b]$ such that at each point $x \in E$, there are two derived numbers $D_1 f(x)$ and $D_2 f(x)$ with

$D_1 f(x) < D_2 f(x)$. Then $\mu^*(E) = 0$.

Proof: Clearly the derived numbers of f are non-negative. For positive rational numbers p,q $(p < q)$ let E_{pq} denote the set of all points x of E such that $D_1 f(x) < p < q < D_2 f(x)$. Then

$$E = \cup_{(p,q)} E_{pq}.$$

Suppose that $\mu^*(E) > 0$. Then for some pair (p,q), $\mu^*(E_{pq}) = \lambda > 0$. Write $A = E_{pq}$. Take any $x \in A$. Then $D_1 f(x) < p$. Then there is a null sequence $\{h_n\}$ $(h_n \neq 0, h_n \in [a, b])$ such that for all n,

$$\frac{|f(x+h_n)-f(x)|}{|h_n|} < p.$$

or $| f(x + h_n) - f(x)| < p|h_n|$ ... ... (8.23)

Let $d(x, h_n) = [x, x + h_n]$ {or $[x + h_n, x]$}.

Denote by $\mathcal{F}$ the family of all closed intervals thus associated with the points of the set A. Then $\mathcal{F}$ covers the set A in the sense of Vitali. Choose any ε with $0 < \varepsilon < \frac{1}{3}\lambda$. Then there exist finite number of pairwisedisjoint closed intervals

$$d(x_1, h_1'), d(x_2, h_2'), \ldots\ldots\ldots\ldots, d(x_r, h_r')$$

in the family $\mathcal{F}$ such that

$$\lambda - \varepsilon < \sum_{i=1}^{r} \mu^*(A \cap d(x_i, h_i')) \le \sum_{i=1}^{r} |h_i| < \lambda + \varepsilon. \quad \ldots \quad (8.24)$$

Write $B = \bigcup_{i=1}^{r} A \cap d^0(x_i, h_i')$, where $d^0(x_i, h_i')$ denotes the open niterval $(x_i, x + h_i')$ or $(x_i + h_i', x)$ Then $B \subset A$ and

$$\mu^*(B) = \sum_{i=1}^{r} \mu^*(A \cap d(x_i h_i')) > \lambda - \varepsilon.$$

Let $G = \bigcup_{i=1}^{r} d^0(x_i, h_i')$. Take any $x \in B$. Then $D_2 f(x) > q$. So there is a null sequence $\{k_n\}$ $(k_n \ne 0, x + k_n \in G)$ such that for all n

$$\frac{|f(x+k_n) - f(x)|}{|k_n|} > q. \quad \ldots \quad (8.25)$$

As above, let $d(x, k_n) = [x, x + k_n]$ {or $[x + k_n, x]$} and $\mathcal{F}^0$ denote the family of all closed intervals thus associted with the points of the set B. Then $\mathcal{F}^0$ covers B in the sense of Vitali. So there exist finite number of pairwise disjoint closed intervals

$$d(y_1, k_1'), d(y_2, k_2'), \ldots, d(y_t, k_t')$$

in the family $\mathcal{F}^0$ such that

$$\mu^*(B) - \varepsilon < \sum_{i=1}^{t} \mu^*(B \cap d(y_i, k_i'))$$

$$\le \sum_{i=1}^{t} |k_i'| < \mu^*(B) + \varepsilon \quad \ldots \quad (8.26)$$

From (8.25) and (8.26) weget

$$\sum_{i=1}^{t} |f(y_i + k_i') - f(y_i)|$$

$$> q \sum_{i=1}^{t} |k_i'| > q\,(\lambda - 2\varepsilon). \qquad \ldots \qquad (8.27)$$

From (8.23) and (8.24) we get

$$\sum_{i=1}^{r} |\, f(x_i+h_i') - f(x_i) \,| < p\,(\lambda + \varepsilon) \qquad \ldots \qquad (8.28)$$

Since each interval $d(y_i, k_i')$ is contained in G, that is, in an interval $d(x_i, h_i')$ and f is increasing it follows that

$$\sum_{i=1}^{t} |\, f(y_i + k_i') - f(y_i)| \le \sum_{i=1}^{r} |\, f(x_i + h_i') - f(x_i)|.$$

Now using (8.27) and (8.28) we get

$$q(\lambda - 2\varepsilon) < p\,(\lambda+\varepsilon)$$

Since $\varepsilon > 0$ is arbitrary we obtain

$$\lambda q \le \lambda\, p \text{ or } q \le p$$

which contradicts the fact that $p < q$. Hence $\mu^*(E) = 0$.

Theorem 8.19. Let $f : [a, b] \to \Omega$ be increasing. Then f possesses finite derivative almost everywhere in $[a, b]$.

Proof : Let E_1 denote the set of all points in $[a, b]$ where f has at least one infinite derived number and E_2 the set of all points x in $[a, b]$ where all the derived numbers of f are finite but $f'(x)$ does not exist. By Lemmas 8.1 and 8.2, $\mu^*(E_1) = 0$ and $\mu^*(E_2) = 0$. If $x \in [a,b] \setminus (E_1 \cup E_2)$, then $f'(x)$ exists and is finite. This proves the result.

Theorem 8.20. Let $f : [a, b] \to \Omega$ be BV on $[a,b]$. Then f possesses finite derivative almost everywhere in $[a, b]$.

Proof : Since f is BV on $[a, b]$, we can express f in the form $f = g - h$, where g and h are increasing on $[a,b]$. Let E_1 and E_2 denote the sets of the points in $[a, b]$ where the functions g and h possess finite derivatives. If $x \in E_1 \cap E_2$, then clearly $f'(x)$ exists finitely. We have

$$\mu^*([a, b] \setminus (E_1 \cap E_2)) \le \mu^*([a,b] \setminus E_1) + \mu^*([a, b] \setminus E_2) = 0.$$

This proves the theorem.

Theorem 8.21. Let $f : [a, b] \to \Omega$ be increasing. Then the derivative f' of f in $[a, b]$ is summable on $[a, b]$ and

$$\int_a^b f'(x)\, dx \le f(b) - f(a).$$

Proof : We define f in $(b, b + 1]$ as follows.

$$f(x) = f(b) \quad \text{for } b < x \le b + 1.$$

Now we define the sequence $\{f_n\}$ of functions on $[a, b]$ as follows.

$$f_n(x) = n\left[f\left(x+\frac{1}{n}\right)-f(x)\right] \text{ for } a \leq x \leq b.$$

Then each f_n is non-negative and measurable on $[a, b]$ and $f_n(x) \to f(x)$ a.e. on $[a,b]$. By Fatou's Lemma

$$\int_a^b f'(x)\,dx \leq \sup\left\{\int_a^b f_n(x)dx\right\} \quad \ldots \quad \ldots \quad (8.29)$$

Now $\int_a^b f_n(x)dx = n\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f(x)dx - n\int_a^b f(x)dx$

$$= n\int_b^{b+\frac{1}{n}} f(x)dx - n\int_a^{a+\frac{1}{n}} f(x)dx$$

$$\leq n\int_b^{b+\frac{1}{n}} f(b)dx - n\int_a^{a+\frac{1}{n}} f(a)dx$$

$$= f(b) - f(a). \quad \ldots \quad \ldots (8.30)$$

From (8.29) and (8.30) we get

$$\int_a^b f'(x)dx \leq f(b) - f(a).$$

Hence f' is summable on $[a, b]$.

Theorem 8.22. Let $f : [a, b] \to \Omega$ be absolutely continuous and $f'(x) = 0$ a.e. on $[a,b]$. Then f is a constant.

Proof : Let α be any point in (a, b).

Denote by E the set of all points x in (a, α) where $f'(x) = 0$. Then $\mu E = \alpha - a$.

Choose any $\varepsilon > 0$. Since f is absolutely continuous on $[a, \alpha]$, there is a $\delta > 0$ such that for every set $\{(a_i, b_i)\}$ of non overlapping intervals in $[a,\alpha]$

$$\sum_i |f(b_i) - f(a_i)| < \varepsilon \text{ when } \sum_i (b_i - a_i) < \delta \quad \ldots \quad (8.31)$$

Let $x \in E$. There is a null sequence $\{h_n\}$ ($h_n \neq 0$, $x + h_n \in (a, \alpha)$) such that

$$|f(x+h_n) - f(x)| < \varepsilon\,|h_n| \quad \ldots \quad \ldots \quad (8.32)$$

Let $d(x, h_n) = [x, x + h_n]$ {or $[x + h_n, x]$}. Denote by $\mathcal{F}$ the family of all closed intervals $d(x, h_n)$ associated with the points of the set E. Clearly the family $\mathcal{F}$ covers the set E in the sense of Vitali. Hence there exist finite number of pairwise disjoint closed intervals $\Delta_1, \Delta_2, \ldots, \Delta_n$ in the family $\mathcal{F}$ such that

$$\sum_{i=1}^{n}\mu(E\cap\Delta_i) > \mu E-\delta=\alpha-a-\delta.$$

Since $\Delta_i \subset (a, \alpha)$ and $\Delta_1, \Delta_2, ..., \Delta_n$ are pairwise disjointi we have

$\sum_{i=1}^{n}\mu(\Delta_i)\leq \alpha - a$. Let $\Delta_i = [a_i, b_i]$ $(i = 1, 2, ..., n)$. Without loss of genera lity we may suppose that $\Delta_1, \Delta_2, ..., \Delta_n$ are in increasing end points. Then $a < a_1 < b_1 < a_2 < b_2 < ... < a_n < b_n < \alpha$.

Since $\sum_{i=1}^{n}\mu\Delta_i \geq \sum_{i=1}^{n}\mu(E\cap\Delta_i) > \alpha - a - \delta,$

we have

$$\sum_{i=0}^{n}(a_{i+1}-b_i) < \delta \; (a = b_0 \text{ and } \alpha = a_{n+1})$$

Now

$$f(a)-f(\alpha) = \{f(b_0)-f(a_1)\}+\{f(a_1)-f(b_1)\}$$
$$+\{f(b_1)-f(a_2)\}+\cdots+\{f(a_n)-f(b_n)\}+\{f(b_n)-f(a_{n+1})\}$$
$$=\sum_{i=0}^{n}\{f(b_i)-f(a_{i+1})\}+\sum_{i=1}^{n}\{f(a_i)-f(b_i)\}$$

So $$|f(a)-f(\alpha)| \leq \sum_{i=0}^{n}|f(b_i)-f(a_{i+1})|+\sum_{i=1}^{n}|f(a_i)-f(b_i)|$$

$$< \varepsilon + \varepsilon.\sum_{i=1}^{n}\mu(\Delta_i) < (1+\alpha-a)\varepsilon.$$

[using (8.31) and (8.32)]

Since $\varepsilon > 0$ is arbitrary, it follows that $f(\alpha) = f(a)$. *Since* α is any point in $(a, b]$, $f(x) = f(a)$ for $a \leq x \leq b$.

Theorem 8.23. Let $f : [a,b] \to \Omega$ be summable on $[a, b]$ and

$\phi(x) = \int_a^x f(t)\,dt$ for $a \leq x \leq b$.

Then $\phi'(x) = f(x)$ *a.e.* on $[a, b]$.

Proof : The function ϕ is absolutely continuous on $[a, b]$. So $\phi'(x)$ exists finitely *a. e.* on $[a, b]$. Let E denote the set of all points in $[a, b]$ where $\phi'(x)$ exists finitely. Then $\mu E = b - a$. Write

$$E' = \{x : x \in E \text{ and } \phi'(x) > f(x)\}$$

and $$E'' = \{x : x \in E \text{ and } \phi'(x) < f(x)\}$$

For any two rational numbers p, q $(q > p)$ let
$E_{pq}=\{x : x \in E'$ and $\phi'(x) > q > p > f(x)\}$.
Then $E' = \cup_{(p,q)} E_{pq}$. Clearly each set E_{pq} is measurable.
If possible, let $\mu(E_{pq}) > 0$ for some pair of rational numbers p,q $(q > p)$.
Write $E_0 = E_{pq}$.
Choose any ε with $0 < \varepsilon < \mu E_0$. Since f is summable on $[a, b]$. there is a $\delta > 0$ such that for every measurable subset e of $[a, b]$ with $\mu e < \delta$,

$$\int_e |f| d\mu < \varepsilon \qquad \ldots \qquad \ldots \qquad (8.33)$$

Let G be an open set with $G \subset (a, b)$ such that $E_0 \subset G$ and
$\mu(G) < \mu E_0 + \delta$. ... ... (8.33$a$)
Take any point $x \in E_0$. There is a null sequnce $\{h_i\}$ $(h_i > 0, x_i + h_i \in G)$ such that

$$\frac{\phi(x+h_i)-\phi(x)}{h_i} > q \qquad \ldots \qquad \ldots \qquad (8.34)$$

Let $d(x, h_i) = [x, x + h_i]$. Denote by $\mathscr{F}$ the family of all closed intervals $d(x, x + h_i)$ associated with the points of the set E_0. Then $\mathscr{F}$ is a Vitali covering of E_0. So there are finite number of pairwise disjoint closed intervals $\Delta_1, \Delta_2, \ldots, \Delta_n$ in the family $\mathscr{F}$ such that

$$\left.\begin{aligned} &\sum_{i=1}^{n} \mu(E_0 \cap \Delta_i) > \mu E_0 - \varepsilon \\ \text{and} \qquad &\sum_{i=1}^{n} \mu\Delta_i < \mu E_0 + \varepsilon \end{aligned}\right\} \qquad \ldots \qquad \ldots \qquad (8.35)$$

Write $\Delta_i = [a_i, b_i]$ and $A = \bigcup_{i=1}^{n} \Delta_i$
From (8.34) and (8.35) we have

$$\int_A f d\mu = \sum_{i=1}^{n} \int_{\Delta_i} f d\mu = \sum_{i=1}^{n} \{\phi(b_i) - \phi(a_i)\}$$

$$> q \sum_{\varepsilon=1}^{n} \mu\Delta_i > q(\mu E_0 - \varepsilon). \qquad \ldots \qquad \ldots \qquad (8.36)$$

On the other hand $A \subset G$ and $A \mid E_0 \subset G \mid E_0$.
Since $G = E_0 \cup (G \setminus E_0)$, $\mu G = \mu E_0 + \mu (G \setminus E_0)$.
So $(A \setminus E_0) \leq \mu (G \setminus E_0) = \mu G - \mu E_0 < \delta$ [using (8.33a)].
Since $f(x) < p$ for all $x \in E_0$ we get using (8.33)

$$\int_A f d\mu = \int_{A \cap E_0} f d\mu + \int_{A \setminus E_0} f d\mu$$

$$< p\,\mu\, E_0 + \varepsilon \qquad \cdots \qquad \cdots \qquad (8.37)$$

Combining (8.36) and (8.37) we have

$$q(\mu E_0 - \varepsilon) < p\mu E_0 + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\mu E_0 = 0$, that is, $\mu\,(E_{pq}) = 0$ and so $\mu E' = 0$.

Similarly we can show that $\mu E'' = 0$. Hence $\phi'(x) = f(x)$ *a.e.* on $[a, b]$.

Theorem 8. 24. Let $f : [a, b] \to \Omega$ be absolutely continuous. Then

$$f(x) = f(a) + \int_a^x f'(t)\, dt \text{ for } a \le x \le b.$$

Proof : Clearly f is BV on $[a, b]$ and hence f' is summable on $[a, b]$. Let

$$\phi(x) = f(a) + \int_a^x f'(t) dt \ (a \le x \le b).$$

Then ϕ is absolutely continuous on $[a, b]$ and $\phi'(x) = f'(x)$ *a.e.* on $[a,b]$.

Let $F(x) = \phi(x) - f(x)$ for $a \le x \le b$. Then F is absolutely continuous on $[a, b]$ and $F'(x) = 0$ *a.e.* on $[a, b]$. So F is *a* constant. We have

$F(a) = \phi(a) - f(a) = f(a) - f(a) = 0.$

So $\qquad F(x) = 0$ for all $x \in [a, b]$, that is, $\phi(x) = f(x)$ for all x in $[a,b]$.

Hence $\qquad f(x) = f(a) + \int_a^x f'(t)dt \ (a \le x \le b).$

Lebesgue points and Lebesgue set.

Let $f : [a, b] \to \Omega$ be summable on $[a, b]$. A point x in (a, b) is said to be *a* Lebesgue point of f if

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)| dt = 0.$$

The set of all Lebesgue points of f in (a, b) is called the Lebesgue set for f in (a, b).

Let f be continuous at $\alpha \in (a, b)$. It is easy to see that α is *a* Lebesgue point of f. The converse is not true.

To show this we consider the following function f defined on $[0, 1]$

$$f(x) = 1 \text{ when } x \text{ is rational}$$
$$= 0 \text{ when } x \text{ is irrational.}$$

Take any irrational point x in $(0,1)$. Choose any $\delta > 0$ with $(x-\delta, x+\delta) \subset (0, 1)$. For any h with $0 < |h| < \delta$, we have

$$\frac{1}{h} \int_x^{x+h} |f(t) - f(x)| dt = 0.$$

So $$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} |f(t)-f(x)|dt = 0.$$

Thus every irrational point in (0,1) is *a* Lebesgue point of *f*. Clearly *f* is not continuous at any *x* in [0, 1]

Theorem 8.25. (Lebesgue). Let $f: [a, b] \to \Omega$ be summable on $[a, b]$. Then almost all points of (a, b) are Lebesgue points of *f*.

Proof: Let *B* denote *a* countable dense subset of Ω. We can take the elements of *B* as

$$\beta_1, \beta_2, \beta_3, \ldots, \beta_n, \ldots$$

Now define the sequence $\{g_n\}$ of functions on $[a, b]$ as follows.

$$g_n(x) = |f(x) - \beta_n| \text{ for } a \le x \le b\ ;\ n = 1, 2, 3, \ldots$$

Clearly each g_n is summable on $[a, b]$.

Let

$$F_n(x) = \int_a^x g_n(t)dt\ (a \le x \le b).$$

Each F_n is absolutely continuous on $[a, b]$ and so $F_n'(x) = g_n(x)$ *a.e.* on $[a,b]$.

Denote by E_n the set of all points in $[a, b]$ such that $F_n'(x) = g_n(x)$ and $E = \bigcap_{n=1}^{\infty} E_n$.

For each n, $\mu([a, b] \mid E_n) = 0$ and so

$$\mu([a, b] \setminus E) \le \sum_{n=1}^{\infty} ([a, b) | E_n) = 0.$$

Take any point $x \in E$ and write $\alpha = f(x)$. Choose any $\varepsilon > 0$. Determine the positive integer n such that $|\beta_n - \alpha| < \frac{1}{3}\varepsilon$.

Let $t \in [a, b]$. We have

$$\big||f(t)-\alpha|-|f(t)-\beta_n|\big| \le |\beta_n - \alpha| < \frac{1}{3}\varepsilon.$$

Since $$\frac{1}{h}\int_x^{x+h} g_n(t)dt = \frac{F_n(x+h)-F_n(x)}{h} \to g_n(x)$$

as $h \to 0$ we can find $\delta > 0$ with $(x - \delta, x + \delta) \subset (a, b)$ such that for any h with $0 < |h| < \delta$,

$$\left|\frac{1}{h}\int_x^{x+h} g_n(t)dt - g_n(x)\right| < \frac{1}{3}\varepsilon.$$

Now for $0 < |h| < \delta$,

$$\left| \frac{1}{h} \int_x^{x+h} |f(t) - \alpha|\, dt - \frac{1}{h} \int_x^{x+h} |f(t) - \beta_n|\, dt \right|$$

$$\leq \left| \frac{1}{h} \int_x^{x+h} |\beta_n - \alpha|\, dt \right| < \frac{1}{3} \varepsilon .$$

and

$$\left| \frac{1}{h} \int_x^{x+h} |f(t) - \alpha|\, dt \right|$$

$$\leq | \; \left| \frac{1}{h} \int_x^{x+h} |f(t) - \alpha|\, dt - \frac{1}{h} \int_x^{x+h} |f(t) - \beta_n|\, dt \right|$$

$$+ \left| \frac{1}{h} \int_x^{x+h} g_n(t)\, dt - g_n(x) \right| + |f(x) - \beta_n|$$

$$< \frac{1}{3} \varepsilon + \frac{1}{3} + \frac{1}{3} \varepsilon = \varepsilon.$$

Hence $$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)|\, dt = 0 .$$

and so x is *a* Lebesgue point of f.

CHAPTER—IX

FOURIER SERIES

Let the function f be summable on $[-\pi, \pi]$. The constants $a_0, a_1, a_2, \ldots;$ $b_1, b_2, b_3, \ldots.$ defined by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx,\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

are known as the Fourier coefficients of f on $[-\pi, \pi]$. The trigonometric series

$$\frac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

is called the Fourier series of f on $[-\pi, \pi]$. We use the following notations

$$S_n (f; x) = \sum_{\nu=1}^{n} (a_\nu \cos \nu_x + b_\nu \sin \nu_x)$$

and $$f(x) \sim \sum_{\nu=1}^{\infty} (a_\nu \cos \nu_x + b_\nu \sin \nu x).$$

Periodic function.

Let the function f be defined on the real line. f is said to be periodic with period w $(w \neq 0)$ if $f(x + w) = f(x)$ for all real numbers x. The least of the periods w (> 0) of f is called the fundamental period or simply the period of f.

Theorem 9.1. Let the function f be periodic with period $2w$ $(w > 0)$ and let it be summable on $[-w, w]$. Then

(i) $\int_{-w+a}^{w+a} f(x)dx = \int_{-w}^{w} f(x)dx$

(ii) $\int_{\alpha+2w}^{\beta+2w} f(x)dx = \int_{\alpha}^{\beta} f(x)dx$

(iii) $\int_{-w}^{w} f(x+a)dx = \int_{-w}^{w} f(x)dx$

(iv) $\int_{a}^{a+2w} f(x)dx = \int_{-w}^{w} f(x)dx.$

Proof : Since f is summable on $[-w, w]$ and periodic with period $2w$, it follows that f is summable on any finite interval.

(i) We have

$$\int_{-w+a}^{w+a} f(x)dx = \int_{-w+a}^{-w} f(x)dx + \int_{-w}^{w} f(x)dx + \int_{w}^{w+a} f(x)dx$$

$$= I_1 + I_2 + I_3 \text{ (say)}$$

In I_3 put $y = x - 2w$. When $x = w$, $y = -w$; $x = w + a$, $y = -w + a$.

So $I_3 = \int_{-w}^{-w+a} f(y+2w)dy = \int_{-w}^{-w+a} f(y)dy$

$$= -\int_{-w+a}^{-w} f(x)dx = -I_1$$

or $\qquad I_1 + I_3 = 0.$

Hence $\int_{-w+a}^{w+a} f(x)dx = \int_{-w}^{w} f(x)dx.$

(ii) Put $y = x - 2w$.

When $x = \alpha + 2w$, $y = \alpha$; $x = \beta + 2w$, $y = \beta$.

So $\qquad \int_{\alpha+2w}^{\beta+2w} f(x)dx = \int_{\alpha}^{\beta} f(y+2w)dy = \int_{a}^{\beta} f(y)dy = \int_{\alpha}^{\beta} f(x)dx$

Other parts are left as exercises.

Theorem 9.2. (Rieman-Lebesgue Theorem).

Let the function f be summable on $[a, b]$. Then

$$\lim_{\lambda\to\infty} \int_a^b f(x)\cos\lambda x dx = 0, \quad \lim_{\lambda\to\infty} \int_a^b f(x)\sin\lambda x dx = 0.$$

Proof : Choose any $\varepsilon > 0$. Since f is summable on $[a,b]$, there is *a* function g continuous on $[a,b]$ (sce Th 7.10) such that

$$\int_a^b |f - g| dx < \frac{1}{3}\varepsilon \qquad \ldots \qquad \ldots \qquad (9.1)$$

Since g is continuous on $[a,b]$ it is uniformly continuous on $[a, b]$. So there is *a* $\delta > 0$ such that

$$|g(x')-g(x'')| < \frac{\varepsilon}{3(b-a)} \qquad \ldots \qquad \ldots \qquad (9.2)$$

for every pair of points x', x'' in $[a, b]$ with $|x'-x''| < \delta$. Take *a* subdivision

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

of $[a, b]$ with $\max(x_i - x_{i-1}) < \delta$. We have

$$\left|\int_a^b f(x)\cos\lambda x dx\right| = \left|\int_a^b \{f(x)-g(x)\}\cos\lambda x dx + \int_a^b g(x)\cos\lambda x dx\right|$$

$$\leq \int_a^b |f(x)-g(x)| dx + \left|\int_a^b g(x)\cos\lambda x dx\right|$$

$$< \frac{1}{3}\varepsilon + \left|\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} g(x)\cos\lambda x dx\right|$$

$$< \frac{1}{3}\varepsilon + \left|\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} \{g(x)-g(x_i)\}\cos\lambda x dx + \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} g(x_i)\cos\lambda x \times dx\right|$$

$$< \frac{1}{3}\varepsilon + \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} |g(x)-g(x_i)| dx + \left|\sum_{i=0}^{n-1} g(x_i)\left[\frac{\sin\lambda x}{\lambda}\right]_{x_i}^{x_{i+1}}\right|$$

$$< \frac{1}{3}\varepsilon + \frac{\varepsilon}{3(b-a)}\cdot\sum_{i=0}^{n-1}(x_{i+1}-x_i) + \sum_{i=0}^{n-1}\frac{2|g(x_i)|}{|\lambda|} \qquad \text{[using (9.2)]}$$

$$\leq \frac{2\varepsilon}{3} + \frac{2nk}{|\lambda|} < \varepsilon \text{ if } |\lambda| > \frac{6nk}{\varepsilon},$$

where $k = \sup\{|g(x)| : a \leq x \leq b\}$.

Hence $\lim\limits_{\lambda\to\infty}\int_a^b f(x)\cos\lambda x\, dx = 0$.

Other part can be similarly proved.

Theorem 9.3. (Bessel's inequality). Let the function f be summable on $[-\pi,\pi]$ and $a_0, a_1, a_2, ...; b_1, b_2, b_3, ...$ be the Fourier coefficients of f on $[-\pi, \pi]$. Then

$$\frac{1}{2}a_0^2 + \sum_{\nu=1}^{\infty}(a_\nu^2 + b_\nu^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx.$$

Proof : If f^2 is not summable on $[-\pi,\pi]$, then there is nothing to prove. So we suppose that f^2 is summable on $[-\pi,\pi]$.

Consider the function $g(x)$ defined by

$$g(x) = [f(x) - S_n(f ; x)]^2$$
$$= f^2(x) - 2 f(x)\, s_n(f ; x) + S_n^2(f ; x).$$

For each n, the functions $S_n(f; x)$ and $S_n^2(f; x)$ are summable on $[-\pi,\pi]$. We have

$$\int_{-\pi}^{\pi} g(x)dx = \int_{-\pi}^{\pi} f^2(x)dx - 2\int_{-\pi}^{\pi} f(x)s_n(f; x)\,dx + \int_{-\pi}^{\pi} S_n^2(f; x)dx \quad \ldots \quad \ldots \quad (9.3)$$

Now

$$\int_{-\pi}^{\pi} f(x)S_n(f; x)dx$$

$$= \frac{1}{2}a_0\int_{-\pi}^{\pi} f(x)dx + \sum_{\nu=1}^{n}\left(a_\nu\int_{-\pi}^{\pi} f(x)\cos\nu x dx + b_\nu\int_{-\pi}^{\pi} f(x)\sin\nu x dx\right)$$

$$= \pi\left\{\frac{1}{2}a_0^2 + \sum_{\nu=1}^{n}\left(a_\nu^2 + b_\nu^2\right)\right\}.$$

and

$$\int_{-\pi}^{\pi} S_n^2(f; x)dx$$

$$= \frac{1}{4}a_0^2\int_{-\pi}^{\pi} dx + a_0\sum_{\nu=1}^{n}\left(a_\nu\int_{-\pi}^{\pi} f(x)\cos\nu x dx + b_\nu\int_{-\pi}^{\pi} f(x)\sin\nu x dx\right)$$

$$+ \sum_{\mu,\nu=1}^{n} a_\mu a_\nu\int_{-\pi}^{\pi}\cos\mu x\cos\nu x dx$$

$$+ 2\sum_{\mu,\nu=1}^{n} a_\mu b_\nu\int_{-\pi}^{\pi}\cos\mu x\sin\nu x dx$$

$$+ \sum_{\mu,\nu=1}^{n} b_\mu b_\nu\int_{-\pi}^{\pi}\sin\mu x\sin\nu x dx$$

$$= \pi\left[\frac{1}{2}a_0^2 + \sum_{\nu=1}^{n}\left(a_\nu^2 + b_\nu^2\right)\right].$$

Substituting in (9.3) we get

$$\int_{-\pi}^{\pi} g(x)dx = \int_{-\pi}^{\pi} f^2(x)dx - \pi\left[\frac{1}{2}a_0^2 + \sum_{\nu=1}^{n}\left(a_\nu^2 + b_\nu^2\right)\right].$$

Since $g(x) \geq 0$, we get

$$\frac{1}{2}a_0^2 + \sum_{\nu=1}^{n}\left(a_\nu^2 + b_\nu^2\right) \le \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx.$$

This gives that

$$\frac{1}{2}a_0^2 + \sum_{\nu=1}^{\infty}\left(a_\nu^2 + b_\nu^2\right) \le \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx.$$

Theorem 9.4. (Dirichlet's Integral). Let the function f be summable on $[-\pi,\pi]$ and be periodic with period 2π. Then for any point x in $[-\pi, \pi]$,

$$S_n(f;\, x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \{f(x+2t)+f(x-2t\}.\frac{\sin(2n+1)t}{\sin t\, dt}dt.$$

Proof : We have

$$a_\nu = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos \nu t dt,\; b_\nu = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin \nu t\, dt.$$

Take any point x in $[-\pi, \pi]$. Then

$$a_\nu \cos \nu x + b_\nu \sin \nu x$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\{\cos \nu x \cos \nu t + \sin \nu x \sin \nu t\}dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\cos \nu(x-t)\, dt$$

$$= \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} f(x+u)\cos \nu du \text{ [Putting } t-x=u]$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)\cos \nu x\, du \; [\because f(x+u)\cos \nu u \text{ is periodic with perio } 2\pi]$$

$$\frac{1}{2}a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+u)\, du \; [\because f \text{ is perodic with period } 2\pi]$$

So,

$$S_n(f;\, x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)\left[\frac{1}{2} + \sum_{\nu=1}^{n}\cos u\right]du$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)\frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}du$$

$$= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x+2t).\frac{\sin(2n+1)t}{\sin t} \text{ [Putting } u = 2t]$$

$$= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{0} f(x+2t).\frac{\sin(2n+1)t}{\sin t}dt + \frac{1}{\pi}\int_{0}^{\frac{\pi}{2}} f(x+2t)\frac{\sin(2n+1)t}{\sin t}dt$$

$$= \frac{1}{\pi}\int_{0}^{\frac{\pi}{2}} f(x-2t)\frac{\sin(2n+1)t}{\sin t}dt + \frac{1}{\pi}\int_{0}^{\frac{\pi}{2}} f(x+2t)\frac{\sin(2n+1)t}{\sin t}dt$$

$$= \frac{1}{\pi}\int_{0}^{\frac{\pi}{2}} \{f(x+2t)+f(x-2t)\}.\frac{\sin(2n+1)t}{\sin t}dt.$$

Theorem 9.5. Let the function f be summable on $[-\pi,\pi]$ and be periodic with period 2π. Then *a* necessary and sufficient condition for the convergence of the Fourier series of f at $x_0 \in [-\pi,\pi]$ to the sum $s(x_0)$ is

$$\lim_{n\to\infty}\frac{1}{\pi}\int_0^\delta \{f(x_0+2t) + f(x_0-2t)-2s(x_0)\}\frac{\sin(2n+1)t}{\sin t}dt = 0,$$

where $0 < \delta < \frac{\pi}{2}$.

Proof : Let $x_0 \in [-\pi, \pi]$ and n be any positive integer. By Dirichlet's Integral we have

$$S_n(f; x_0) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \{f(x_0+2t)+f(x_0-2t)\}\frac{\sin(2n+1)t}{\sin t}dt.$$

By actual evaluation we get

$$\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)t}{\sin t} dt = \frac{\pi}{2}.$$

So

$$s(x_0) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} s(x_0).\frac{\sin(2n+1)t}{\sin t} dt.$$

Therefore

$$S_n(f\,;\,x_0) - s(x_0) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \{f(x_0+2t)+f(x_0-2t)-2s(x_0)\} \frac{\sin(2n+1)t}{\sin t} dt.$$

Write $\phi(t) = f(x_0 + 2t) + f(x_0 - 2t) - 2\, s(x_0)$.

Clearly ϕ is summable on $\left[0, \frac{\pi}{2}\right]$. Take any δ with $0 < \delta < \frac{\pi}{2}$. We have

$$S_n(f\,;\,x_0) - s(x_0) = \frac{1}{\pi} \int_0^{\delta} \phi(t) \frac{\sin(2n+1)t}{\sin t} dt + \frac{1}{\pi} \int_{\delta}^{\pi} \phi(t).\frac{\sin(2n+1)t}{\sin t} dt. \quad \ldots \quad \ldots \quad (9.4)$$

By Riemann-Lebesgue Theorem we get

$$\lim_{n\to\infty} \frac{1}{\pi} \int_{\delta}^{\frac{\pi}{2}} \phi(t).\frac{\sin(2n+1)t}{\sin t} dt = 0 \quad \ldots \quad \ldots \quad (9.5)$$

Suppose that

$$\lim_{n\to\infty} \frac{1}{\pi} \int_0^{\delta} \phi(t).\frac{\sin(2n+1)t}{\sin t} dt = 0 \quad \ldots \quad \ldots \quad (9.6)$$

From (9.4), (9.5) and (9.6) we see that $S_n\,(f\,;\,x_0) \to s(x_0)$ as $n \to \infty$. Therefore the Fourier series of f at x_0 converges to the sum $s(x_0)$.

Next, suppose that the Fourier series of f at x_0 converges to the sum $s(x_0)$. Then $S_n(f\,;\,x_0) \to s(x_0)$ as $n \to \infty$. Hence from (9.4) and (9.5) it follows that (9.6) holds.

Corollary 9.5.1. Let the function f be summable on $[-\pi, \pi]$ and periodic

with period 2π. Then *a* necessary and sufficient condition for the convergence of the Fourier series of f at x_0 to the sum $s(x_0)$ is

$$\lim_{n\to\infty} \frac{1}{\pi}\int_0^\delta \{ f(x_0+2t) + f(x_0 + f(x_0-2t) - 2\, s(x_0)\} \frac{\sin(2n+1)t}{t} dt = 0.$$

where δ is any positive number with $0 < \delta < \frac{\pi}{2}$.

Proof : Take any δ with $0 < \delta < \frac{\pi}{2}$. Write

$\phi(t) = f(x_0+2t) + f(x_0 - 2t) - 2S(x_0)$

and $\psi(t) = \frac{1}{t} - \frac{1}{\sin t}$ for $0 < t \le \delta$

$= 0$ for $t = 0$.

Since $\psi(t) \to 0$ as $t \to 0+$, it follows that ψ is continuous on $[0,\delta]$. So $\phi\,\psi$ is summable on $[0, \delta]$. By Riemann-Lebesgue Theorem

$$\lim_{n\to\infty} \frac{1}{\pi}\int_0^\delta \phi(t)\, \psi(t) \sin(2n + 1)\, t\, dt = 0$$

or $$\lim_{n\to\infty} \frac{1}{\pi}\int_0^\delta \phi(t)\left[\frac{1}{t} - \frac{1}{\sin t}\right] \sin(2n+1)\, t dt = 0$$

or $$\lim_{n\to\infty} \frac{1}{\pi}\int_0^\delta \phi(t).\frac{\sin(2n+1)t}{t} dt = \lim_{n\to\infty} \frac{1}{\pi}\int_0^\delta \phi(t)\frac{\sin(2n+1)t}{\sin t} dt.$$

Hence the result follows form Theorem 9.5.

Theorem 9.6. (Riemann's Theorem). Let the function f be summatle on $[-\pi,\pi]$ and $f(x + 2\pi) = f(x)$ for all real x. The behaviour of the Fourier series of f at *a* point $x_0 \in [-\pi,\pi]$ regarding its convergence depends only on those values of f which it takes in the immediate neighbourhood of x_0.

Proof : *A* necessary and sufficient condition for the convergence of the Fourier series $s(f\,;\, x_0)$ of f at x_0 to the sum $s(x_0)$ is

$$\lim_{n\to\infty} \frac{1}{\pi}\int_0^\delta \{ f(x_0+2t)+f(x_0 - 2t) - 2s(x_0)\}.\ \frac{\sin(2n+1)t}{\sin t} dt = 0,$$

where δ is any positive number with $0 < \delta < \frac{\pi}{2}$.

The integral here involves only the values of $f(x_0 \pm 2t)$ for $t \in [0, \delta]$, that is, the values of $f(x)$ for x in $(x_0-2\delta, x_0+2\delta)$. Since δ is arbitrary it follows that the behaviour of $s(f; x_0)$ regarding its convergence depends only on those values of f which it takes in the immediate neighbourhood of x_0.

Theorem 9.7. Let function ϕ be increasing on $[0, \delta]$ and let

$\lim\limits_{\lambda\to 0+} \phi(t) = \phi_0$. Then

$$\lim_{\lambda\to+\infty} \frac{2}{\pi}\int_0^{\delta}\phi(t).\frac{\sin\lambda t}{t}dt = \phi_0.$$

Proof : Let $\psi(t) = \phi(t) - \phi_0$ for $0 < t \le \delta$ and $\psi(0) = 0$. Then ψ is bounded,

non-negative and increasing on $[0, \delta]$ and $\lim\limits_{b\to 0+} \psi(t) = 0$.

Since the integral $\int_0^{\infty}\frac{\sin x}{x}dx$ is convergent, there is *a* positive number k such that for $0 \le a < b$,

$$\left|\int_a^b\frac{\sin x}{x}dx\right| \le k. \quad \text{... ...} \quad (9.7)$$

Choose any $\varepsilon > 0$. We can find *a* positive number δ' $(0 < \delta' < \delta)$ such that

$$0 \le \psi(t) < \frac{\varepsilon}{3k} \text{ for } 0 < t \le \delta' \quad \text{... ...} \quad (9.8)$$

Clearly the fumction $\frac{\phi(t)}{t}$ is bounded and measurable on $[\delta', \delta]$. So by Riemann Lebesgue Theorem, there is *a* positive number λ_1 such that for $\lambda \ge \lambda_1$

$$\left|\frac{2}{\pi}\int_{\delta'}^{\delta}\phi(t)\frac{\sin\lambda t}{t}dt\right| < \frac{1}{3}\varepsilon \quad \text{... ...} \quad (9.9)$$

Take any $\lambda > 0$. We have

$$\frac{2}{\pi}\int_0^{\delta}\phi(t)\frac{\sin\lambda t}{t}dt =$$

$$= \frac{2}{\pi}\int_0^{\delta'}\phi(t)\frac{\sin\lambda t}{t}dt + \frac{2}{\pi}\int_{\delta'}^{\delta}\phi(t).\frac{\sin\lambda t}{t}dt.$$

$$= \frac{2}{\pi}\int_0^{\delta'} \phi_0 . \frac{\sin \lambda t}{t} dt + \frac{2}{\pi}\int_0^{\delta'} \psi(t). \frac{\sin \lambda t}{t} dt + \frac{2}{\pi}\int_{\delta'}^{\delta} \phi(t) \frac{\sin \lambda t}{t} dt$$

$$= I_1(\lambda) + I_2(\lambda) + I_3(\lambda) \text{ (say)} \qquad \ldots \qquad (9.10)$$

Now $\lim\limits_{\lambda\to\infty} I_1(\lambda) = \lim\limits_{\lambda\to+\infty} \frac{2}{\pi}\phi_0 \int_0^{\lambda\delta'} \frac{\sin t}{t} dt$

$$= \frac{2}{\pi}\phi_0 \int_0^{\infty} \frac{\sin t}{t} dt = \phi_0 .$$

So we can find *a* positive number λ_0 $(\lambda_0 \geq \lambda_1)$ such that for $\lambda \geq \lambda_0$

$$|I_1(\lambda) - \phi_0| < \frac{1}{3}\varepsilon \qquad \ldots \qquad \ldots \qquad (9.11)$$

By second Mean value Theorem we get

$$I_2(\lambda) = \frac{2}{\pi}\psi(\delta') \int_{\eta}^{\delta'} \frac{\sin \lambda t}{t} dt,$$

where $0 \leq \eta \leq \delta'$.

$$= \frac{2}{\pi}\psi(\delta') \int_{\lambda\eta}^{n\delta'} \frac{\sin t.}{t} dt.$$

So using (9.7) and (9.8) we get

$$|I_2(\lambda)| \leq \frac{2}{\pi}.\psi(\delta')k$$

$$< \frac{2}{\pi}.\frac{\varepsilon}{3k}.k < \frac{1}{3}\varepsilon. \qquad \ldots \qquad \ldots \qquad (9.12)$$

Using (9.9) we get for $\lambda \geq \lambda_0$

$$|I_3(\lambda)| < \frac{1}{3}\varepsilon. \qquad \ldots \qquad \ldots \qquad (9.13)$$

Combining (9.11), (9.12) and (9.13) we obtain

$$\left|\frac{2}{\pi}\int_0^{\delta} \phi(t) \frac{\sin \lambda t}{t} dt - \phi_0\right| < \varepsilon \text{ when } \lambda \geq \lambda_0.$$

Therefore,

$$\lim_{\lambda\to\infty} \frac{2}{\pi}\int_0^{\delta} \phi(t) \frac{\sin \lambda t}{t} dt = \phi_0.$$

Note. 9.1. It is easy to see that the result is also true when ϕ is decreasing.

Corollary 9.7.1. Let the function ϕ be bounded and increasing on $[0.\delta]$.

and let $\lim\limits_{t \to 0+} \phi(t) = \phi_0$. Then

$$\lim_{\lambda \to +\infty} \frac{2}{\pi} \int_0^\delta \phi(t) \frac{\sin \lambda t}{\sin t} dt = \phi_0 .$$

Proof : Let $\psi(t) = \dfrac{1}{\sin t} - \dfrac{1}{t}$ for $0 < t \leq \delta$ and $\psi(0) = 0$. Sinee $\psi(t) \to 0$ as $t \to 0+$, it follows that ψ is continuous on $[0, \delta]$ and so ψ is summable on $[0, \delta]$.

We have

$$\frac{2}{\pi} \int_0^\delta \phi(t) . \frac{\sin \lambda t}{\sin t} dt$$

$$= \frac{2}{\pi} \int_0^\delta \phi(t) . \frac{\sin \lambda t}{t} dt + \frac{2}{\pi} \int_0^\delta \phi(t) \left[\frac{1}{\sin t} - \frac{1}{t} \right] \sin \lambda\, t\, dt.$$

$$= \frac{2}{\pi} \int_0^\delta \phi(t) . \frac{\sin \lambda t}{t} dt + \frac{2}{\pi} \int_0^\delta \phi(t) \psi(t) \sin \lambda\, t\, dt$$

$= I_1(\lambda) + I_2(\lambda)$ (say).

Clearly $\phi(t)\, \psi(t)$ is summable on $[0, \delta]$. So by Riemann-Lebesgue Theorem $I_2(\lambda) \to 0$ as $\lambda \to +\infty$ and by Theorem 9.7, $I_1(\lambda) \to \phi_0$ as $\lambda \to +\infty$.

Hence

$$\lim_{\lambda \to +\infty} \frac{2}{\pi} \int_0^\delta \frac{\sin \lambda t}{\sin t} . \phi(t) = \phi_0 .$$

Theorem 9.8. (Jordan's Test). Let the function f be summable on $[-\pi, \pi]$ and be periodic with period 2π. If f is BV on some neighbourhood of the point $x_0 \in [-\pi, \pi]$, the Fourier series of f at x_0 converges to $\frac{1}{2} [f(x_0 + 0) + f(x_0 - 0)]$.

Proof : Suppose that f is BV on the neighbourhood $[x_0 - \delta_0, + \delta_0]$ of the point x_0 $(0 < \delta_0 < \frac{\pi}{2})$. Then $f(x_0 + 0)$ and $f(x_0 -)$ exist finitely.

Take $s(x_0) = \frac{1}{2}[f(x_0+0) + f(x_0 - 0)]$

and $\phi(t) = f(x_0 + 2t) + f(x_0-2t) -2s(x_0)$.

Clearly ϕ is BV on $[0, \delta]$, where $\delta = \frac{1}{2}\delta_0$. and $\phi(t) \to 0$ as $t \to 0+$. So we can express ϕ in the form

$$\phi(t) = \phi_1(t) - \phi_2(t) \text{ for } 0 \le t \le \delta.$$

where ϕ_1, ϕ_2 are increasing and non-negative on $[0,\delta]$ and $\phi_1(t) \to 0$, $\phi_2(t) \to 0$ as $t \to 0+$.

$$S_n(f; x_0) - s(x_0) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \phi(t).\frac{\sin(2n+1)t}{\sin t}dt$$

$$= \frac{1}{\pi}\int_0^{\delta}\phi_1(t)\frac{\sin(2n+1)t}{\sin t}dt - \frac{1}{\pi}\int_0^{\delta}\phi_2(t)\frac{\sin(2n+1)t.}{\sin t}$$

$$+\frac{1}{\pi}\int_{\delta}^{\frac{\pi}{2}} \phi(t).\frac{\sin(2n+1)t}{\sin t}dt$$

$$= I_1(n) - I_2(n) + I_3(n) \text{ (say).}$$

Since $\frac{1}{\sin t}$ is continuous on $\left[\delta, \frac{\pi}{2}\right]$, the function $\frac{\phi(t)}{\sin t}$ is summable on $\left[\delta, \frac{\pi}{2}\right]$. So by Riemann-Lebesgue Theorem, $I_3(n) \to 0$. as $n \to \infty$.

By corollary 9.7.1. $I_1(n) \to 0$, $I_2(n) \to 0$ as $n \to \infty$.

So $S_n(f; x_0) \to s(x_0)$ as $n \to \infty$. Therefore the Fourier series of f at x_0 converges to the sum $\frac{1}{2}[f(x_0+0) + f(x_0 - 0)]$.

Corollary 9.8.1. Let the function f be BV on $[-\pi,\pi]$ and periodic with period 2π. Then the Fourier series of f at each point of continuity of f converges to the value of the function, that is,

$$f(x) = \frac{1}{2}a_0 + \sum_{\nu=1}^{\infty}(a_\nu \cos \nu x + b_\nu \sin \nu x)$$

at each point of continuity of f.

Theorem 9.9. (Dini's Test). Let the function f be summable on $[-\pi,\pi]$ and periodic with period 2π and let $x_0 \in [-\pi, \pi]$. If there is a real number $s(x_0)$ such that the function

$$\frac{1}{t}\,[f(x_0+2t) + f(x_0-2t) - 2s\,(x_0)]$$

is summable on $[0, \delta]$ (where $0 < \delta < \frac{\pi}{2}$), then the Fourier series of f at x_0 converges to $s(x_0)$.

Proof : Write $\phi(t) = f(x_0+2t) + f(x_0-2t) - 2s(x_0)$.

Since $\frac{\phi(t)}{t}$ is summable on $[0,\delta]$ $(0 < \delta < \frac{\pi}{2})$, by Riemann-Lebesgue Theorem

$$\lim_{n\to\infty} \frac{1}{\pi}\int_0^{\delta} \phi(t)\frac{\sin(2n+1)t}{t}\,dt = 0 .$$

Hence by Theorem 9.7, the Fourier series of f at x_0 converges to the sum $s(x_0)$.

Theorem 9.10. Let the function f be summable on $[-\pi,\pi]$ and periodic with period 2π. If f possesses finite left and right hand derivatives at $x_0 \in [-\pi,\pi]$, then the Fourier series of f at x_0 converges to $f(x)$.

Proof : For $0 < t \leq \frac{\pi}{2}$, write

$$\left.\begin{aligned} \frac{f(x_0+2t)-f(x_0)}{2t} - f'_+(x_0) = \eta_1(t) \\ \frac{f(x_0-2t)-f(x_0)}{-2t} - f'_-(x_0) = \eta_2(t) \end{aligned}\right\} \quad \cdots \quad \cdots \quad (9.14)$$

Since $f'_+(x_0)$ and $f'_-(x_0)$ exist finitely $\eta_1(t) \to 0$, $\eta_2(t) \to 0$ as $t \to 0+$. So we can find a δ $(0 < \delta < \frac{\pi}{2})$ such that $\eta_1(t)$ and $\eta_2\,(f)$ are bounded on $(0, \delta]$. Clearly $\eta_1(t)$ and $\eta_2\,(f)$ are measurable on $[0, \delta]$. From (9.14) we have

$$f(x_0 + 2t) - f(x_0) = 2t\,[f'_+(x_0) + \eta_1(t)],$$

$$f(x_0-2t) - f(x_0) = -\,2t\,[f'_-(x_0) + \eta_2\,(f)].$$

So $\phi(t) = f(x_0 + 2t) + f(x_0 - 2t) - 2f(x_0)$

$$= 2t\,[f'_+(x_0) - f'_-(x_0) + \eta_1\,(f) - \eta_2(f)]$$

or $\dfrac{\phi(t)}{t} = 2[f'_+(x_0) - f'_-(x_0) + \eta_1(t) - \eta_2(t)]$.

Clearly $\dfrac{\phi(t)}{t}$ is bounded and measurable on $(0, \delta)$ which gives that $\dfrac{\phi(t)}{t}$ is summable on $[0, \delta]$. Hence by Dini's Test, the Fourier series of f at x_0 converges to $f(x_0)$.

Theorem 9.11. (De la vallee-Poussin's Test).

Let the function f be summable on $[-\pi,\pi]$ and periodic with period 2π and let $x_0 \in [-\pi,\pi]$. If there is *a* real number $s(x_0)$ such that the function

$$\psi(t) = \frac{1}{t}\int_0^t [f(x_0 + 2u) + f(x_0-2u)-2s(x_0)]du$$

is BV on $[0,\delta]$ $(0 < \delta < \frac{\pi}{2})$ and $\psi(t) \to 0$ as $t \to 0+$, then the Fourier series of f at x_0 converges to the sum $s(x_0)$.

Proof : Write $\phi(t) = f(x_0 + 2t) + f(x_0 - 2t) - 2s(x_0)$. We have

$$t\psi(t) = \int_0^t \phi(u)du$$

So $\phi(t) = \psi(t) + t\psi'(t)$ *a.e.* in $[0,\delta]$.

Since ψ is BV on $[0, \delta]$, $\psi'(t)$ is summable on $[0, \delta]$. We have

$$I_n = \int_0^\delta \phi(t)\frac{\sin(2n+1)t}{\sin t}dt$$

$$= \int_0^\delta \psi(t)\frac{\sin(2n+1)t}{\sin t}dt + \int_0^\delta \psi'(t).\frac{t}{\sin t}.\sin(2n+1)t\,dt$$

$$= I'_n + I''_n \text{ (say)}$$

Since ψ is BV on $[0,\delta]$ and $\psi(t) \to 0$ *as* $t \to 0+$, it follows that $I'_n \to 0$ as $n \to \infty$. [see cor. 9.7.1] Again, $\dfrac{t}{\sin t}$ is continuous on $(0,\delta]$ and $\dfrac{t}{\sin t} \to 1$ as $t \to 0+$. This gives that $\dfrac{t}{\sin t}$ is bounded on $(0, \delta)$. Since $\psi'(t)$ is summable on $[0,\delta]$, $\psi'(t).\dfrac{t}{\sin t}$ is summable on $[0,\delta]$. So. $I''_n \to 0$ as $n \to \infty$. Hence $I_n \to 0$ as $n \to \infty$. This proves the Theorem.

Dirichlet's condition.

A function f defined on $[a, b]$ is said to satisfy Dirichlet's condition on $[a, b]$ if one of the following holds.

(1) f is of bounded variation on $[a, b]$.

(2) f is summable on $[a,b]$, f has finite number of infinite discontinuities in $[a,b]$ and if these finite number of discontinuities be excluded by arbitrary small neighbourhoods, f is BV on the remaing subintervals.

Theorem 9.12. (Fourier Theorem) Let the function f satisfy Dirichlet's condition on $[-\pi, \pi]$ and be periodic with period 2π and let $x_0 \in [-\pi,\pi]$. If $f(x_0+0)$ and $f(x_0-0)$ exist finitely then the Fourier series of f at x_0 converges to the sum $\frac{1}{2}[f(x_0+0) + f(x_0-0)]$ and at $\pm\pi$ to the sum

$$\frac{1}{2}[f(\pi-0) + f(-\pi+0].$$

Proof : Since $f(x_0+0)$ and $f(x_0-0)$ exist finitely f is bounded in some neighbourhood of x_0. Again, since f satisfies Derichlèt's condition on $[-\pi, \pi]$ it follows that f is BV on $[x_0-\delta, x_0+\delta]$ for some δ with $0 < \delta < \frac{\pi}{2}$, Therefore by Jordan's Test, the Fourier series of f at x_0 converges to the sum $\frac{1}{2}[f(x_0+0) + f(x_0-0)]$. Suppose that $f(\pi+0)$ and $f(\pi-0)$ exist finitely. The Fouries Series at π converges to the sum $\frac{1}{2}[f(\pi-0)+f(\pi+0)]$. Since $f(-\pi+t) = f(\pi+t)$ for all t. $f(-\pi+0) = f(\pi+0)$. Hence the Fourier series at π converges to $\frac{1}{2}[f(\pi-0) + f(-\pi+0)]$. Similarly Fourier series at $-\pi$ converges to $\frac{1}{2}[f(\pi-0) + f(-\pi+0)]$.

Theorem 9.13. Let the function f be summable on $[-\pi,\pi]$ and periodic with perodic with period 2π. If

$$f(x) \sim \frac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b\nu \sin \nu x),$$

then for all x in $[-\pi, \pi]$

$$\int_0^x f(t)dt = \frac{1}{2}a_0\, x + \sum_{\nu=1}^{\infty}\int_0^x (a_\nu \cos \nu t + b\nu \sin \nu t)\, dt.$$

Further if f is square summable on $[-\pi,\pi]$, then the series on the right converges uniformly on $[-\pi,\pi]$.

Proof : We define the function ϕ on the real line as follows.

$$\phi(x) = \int_0^x \left\{ f(t) - \frac{1}{2}a_0 \right\} dt \text{ for} -\pi \le x \le \pi$$

and $\phi(x+2\pi) = \phi(x)$ for all real x.

Then ϕ is absoutely continuous and so BV on $[-\pi,\pi]$. Hence by Jordan's Test

$$\phi(x) = \frac{1}{2}A_0 + \sum_{\nu=1}^{\infty} (A_\nu \cos \nu x + B_\nu \sin \nu x), \quad \ldots \quad (9.15)$$

for all x in $[-\pi,\pi]$, where the series on the right is the Fourier series of ϕ on $[-\pi,\pi]$.

For any $\nu \ge 1$, we have

$$A_\nu = \frac{1}{\pi}\int_{\pi}^{\pi} \phi(x) \cos \nu x dx$$

$$= \frac{1}{\pi}\left[\phi(x)\frac{\sin \nu x}{\nu} \right]_{-\pi}^{\pi} - \frac{1}{\pi\nu}\int_{-\pi}^{\pi}\left\{ f(x) - \frac{1}{2}a_0 \right\} \sin \nu x dx$$

$$= -\frac{1}{\pi\nu}\int_{-\pi}^{\pi} f(x) \sin \nu x + \frac{a_0}{\pi\nu}\int_{-\pi}^{\pi} \sin \nu x dx$$

$$= -\frac{b\nu}{\nu}.$$

$$B_\nu = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(x) \sin \nu x\, dx$$

$$= \frac{1}{\pi}\left[\phi(x)\frac{\cos \nu x}{-\nu} \right]_{-\pi}^{\pi} + \frac{1}{\pi\nu}\int_{-\pi}^{\pi}\left\{ f(x) - \frac{1}{2}a_0 \right\} \cos \nu x dx$$

$$= -\frac{1}{\pi\nu}[\phi(\pi) \cos \nu\pi - \phi(-\pi) \cos \nu\pi]$$

$$+\frac{1}{\pi\nu}\int_{-\pi}^{\pi} f(x)\cos \nu x\, dx - \frac{a_0}{\pi\nu}\int_{-\pi}^{\pi}\cos \nu x dx$$

$$= \frac{a_\nu}{\nu} \quad [\because \phi(\pi) = \phi(-\pi)]$$

Putting $x = 0$ in (9.15) we get

$$0 = \frac{1}{2}A_0 + \sum_{\nu=1}^{\infty} A_\nu$$

or
$$\frac{1}{2}A_0 = -\sum_{\nu=1}^{\infty} A_\nu = \sum_{\nu=1}^{\infty}\frac{b_\nu}{\nu} \qquad \ldots \qquad \ldots \qquad (9.16)$$

For any x in $[-\pi, \pi]$ we have

$$\int_0^x (a_\nu \cos \nu t + b_\nu \sin \nu t)\, dt$$

$$= a_\nu \frac{\sin \nu t}{\nu} - b_\nu \frac{\cos \nu t}{\nu}\Bigg]_0^x$$

$$= \frac{a_\nu}{\nu}\sin \nu x - \frac{b_\nu}{\nu}\cos \nu x + \frac{b_\nu}{\nu}.$$

So
$$\sum_{\nu=1}^{\infty}\int_0^x (a_\nu \cos \nu t + b_\nu \sin \nu t)\, dt$$

$$= \left(\sum_{\nu=1}^{\infty}\frac{b_\nu}{\nu}\right) + \sum_{\nu=1}^{\infty}(A_\nu \cos \nu x + B_\nu \sin \nu x)$$

$$= \frac{1}{2}A_0 + \sum_{\nu=1}^{\infty}(A_\nu \cos \nu x + B_\nu \sin \nu x).$$

$$= \phi(x) = \int_0^x \left\{ f(t) - \frac{1}{2}a_0 \right\} dt$$

$$= \int_0^x f(t)dt - \frac{1}{2}a_0 x.$$

Hence

$$\int_0^x f(t)dt = \frac{1}{2}a_0 x + \sum_{\nu=1}^{\infty} \int_0^x (a_\nu \cos \nu t + b\nu \sin \nu t)dt.$$

Now suppose that f is square summable on $[-\pi, \pi]$. By Bessel's inequality we get

$$\frac{1}{2}a_0^2 + \sum_{\nu=1}^{\infty}\left(a_\nu^2 + b_\nu^2\right) \le \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx \qquad \dots \qquad \dots \qquad (9.17)$$

Choose any $\varepsilon > 0$. From (9.16) and (9.17) we see that the series $\sum \frac{b_\nu}{\nu}$ and $\sum\left(a_\nu^2 + v_\nu^2\right)$ are convergent. So we can find *a* positive integer n_0 such that for $m > n \ge n_0$

$$\left.\begin{aligned} &\sum_{\nu=n}^{m}\left(a_\nu^2 + v_\nu^2\right) < \varepsilon \\ &\left|\sum_{\nu=n}^{m} \frac{b_\nu}{\nu}\right| < \varepsilon \\ &\sum_{\nu=n}^{m} \frac{1}{\nu^2} < \varepsilon \end{aligned}\right\} \qquad \dots \qquad \dots \qquad (9.18)$$

Take any two positive integers m, n with $m > n \ge n_0$ and any x in $[-\pi, \pi]$, Then

$$\left|\sum_{\nu=n}^{m} \int_0 \left(a_\nu \cos \nu t + b_\nu \sin \nu t\right)dt\right|$$

$$\le \sum_{\nu=n}^{m}\left\{\left|a_\nu\right| \frac{|\sin \nu x|}{\nu} + \left|b_\nu\right| . \frac{|\cos \nu x|}{\nu}\right\} + \left|\sum_{\nu=n}^{m} \frac{b\nu}{\nu}\right|$$

$$\le \sum_{\nu=n}^{m}\left(a_\nu^2 + b_\nu^2\right)^{\frac{1}{2}}\left(\sum_{\nu=n}^{m} \frac{1}{\nu^2}\right)^{\frac{1}{2}} + \left|\sum_{\nu=n}^{m} \frac{b_\nu}{\nu}\right|$$

$< \varepsilon + \varepsilon = 2\,\varepsilon$ [using Caucby's inequality] and (9.18)

Hence the series

$$\frac{1}{2}a_0x+\sum_{\nu=1}^{\infty}\int_0^x(a_\nu \cos \nu t+b_\nu \sin \nu t s)dt$$

converges uniformly on $[-\pi,\pi]$.

Corollary 9.13.1. If the function f is summable on $[-\pi,\pi]$ and $f(x + 2\pi) = f(x)$ for all real x, then the series $\sum \frac{b_n}{n}$ converges, where

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx dx.$$

The result follows from (9.16).

Lemma 9.1. For all real number x, the series

$$\sum_{\nu=1}^{\infty}\frac{\sin \nu x}{\nu} \qquad \dots \qquad \dots \qquad (9.19)$$

converges and

$$\left|\sum_{\nu=1}^{n}\frac{\sin \nu x}{\nu}\right| < 2\sqrt{\pi}$$

for all real x and any positive integer n.

Proof : Take any x in $(0, \pi)$. For any positive integer n, let

$$S_n(x) = \sum_{\nu=1}^{n}\frac{\sin \nu x}{\nu}.$$

Choose any two positive integers m, n with $m > n$. We have

$$|S_m(x) - S_n(x)| = \left|\sum_{\nu=n+1}^{m}\frac{\sin \nu x}{\nu}\right|$$

$$=\frac{1}{2}\operatorname{cosec}\frac{x}{2}\left|\sum_{\nu=n+1}^{m}\frac{2\sin \nu x \sin\frac{x}{2}}{\nu}\right|$$

$$=\frac{1}{2}\operatorname{cosec}\frac{x}{2}\left|\sum_{\nu=n+1}^{m}\frac{1}{\nu}\left\{\cos\left(\nu-\frac{1}{2}\right)x-\cos\left(\nu+\frac{1}{2}\right)x\right\}\right|$$

$$= \frac{1}{2}\operatorname{cosec}\frac{x}{2}\left|\frac{\cos\left(n+\frac{1}{2}\right)x}{n+1}+\sum_{\nu=n+1}^{m-1}\left(\frac{1}{\nu+1}-\frac{1}{\nu}\right)\cos\left(\nu+\frac{1}{2}\right)x-\frac{\cos\left(n+\frac{1}{2}\right)x}{m}\right|$$

$$\leq \frac{1}{2}\operatorname{cosec}\frac{x}{2}\left\{\frac{1}{n+1}+\sum_{\nu=n+1}^{m-1}\left(\frac{1}{\nu}-\frac{1}{\nu+1}\right)+\frac{1}{m}\right\}$$

$$= \frac{\operatorname{cosec}\frac{x}{2}}{n+1} \to 0 \text{ as } m, n \to \infty.$$

Therefore the series (9.19) converges.

Now let $x \in (0, \pi)$. Take any positive integer n. If $n \leq \frac{\sqrt{\pi}}{x}$, then

$$|S_n(x)| \leq \sum_{\nu=1}^{n}\frac{|\sin \nu x|}{\nu} \leq nx \leq \sqrt{\pi} < 2\sqrt{\pi}.$$

Suppose that $n > \frac{\sqrt{\pi}}{x}$. We can find a positive integer q with

$$q \leq \frac{\sqrt{\pi}}{x} < q+1.$$

We have

$$|S_n(x) - S_q(x)| \leq \frac{\operatorname{cosec} x/2}{q+1} < \frac{2}{\sqrt{\pi}}\left(\frac{x/2}{\sin x/2}\right) \leq \frac{2}{\sqrt{\pi}}\frac{\pi}{2} = \sqrt{\pi}.$$

or $|S_n(x)| \leq |S_n(x) - S_q(x)| + |S_q(x)| < \sqrt{\pi} + \sqrt{\pi} = 2\sqrt{\pi}$.
Since $S_n(0) = S_n(\pi) = 0$ and $S_n(-x) = -S_n(x)$, the proof is complete.

Lemma 9.2. For $0 \leq a < b \leq \frac{\pi}{2}$

$$\left|\int_a^b \frac{\sin(2n+1)x}{\sin x}dx\right| < \left(4\sqrt{\pi}+\frac{\pi}{2}\right).$$

Proof : Let a and b be two real numbers with $0 \le a < b \le \dfrac{\pi}{2}$. We have for any real x.

$$\frac{\sin(2n+1)x}{\sin x} = 1 + 2\sum_{v=1}^{n} \cos 2vx.$$

So
$$\int_a^b \frac{\sin(2n+1)x}{\sin x}\,dx = (b-a) + \sum_{v=1}^{n} \frac{\sin 2vb - \sin 2va}{v}.$$

Therefore

$$\left|\int_a^b \frac{\sin(2n+1)x}{\sin x}\,dx\right| \le (b-a) + \left|\sum_{v=1}^{n} \frac{\sin 2vb}{v}\right| + \left|\sum_{v=1}^{n} \frac{\sin 2va}{v}\right|$$

$$\le \frac{\pi}{2} + 2\sqrt{\pi} + 2\sqrt{\pi} = \frac{\pi}{2} + 4\sqrt{\pi}.$$

Theorem 9.14. Let the function f be BV and continuous on $[-\pi, \pi]$ and be periodic with period 2π. Then the Fourier series of f on $[-\pi, \pi]$ converges uniformly to f on $[-\pi, \pi]$.

Proof : Clearly f is BV and continuous on any finite interval ; in particular on $[-2\pi, 2\pi]$. By Jordan's Test we get

$f(x) = \dfrac{1}{2}a_0 + \displaystyle\sum_{v=1}^{\infty} (a_v \cos vx + b_v \sin vx)$ for all real x, where the series on the right is the Fourier series of f on $[-\pi, \pi]$. Take any x in $[-\pi,\pi]$ and any positive integer n. Then

$$S_n(f\,;\,x) - f(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \phi(x,t).\frac{\sin(24+1)t}{\sin t}\,dt$$

where $\phi(x,t) = f(x+2t) + f(x-2t) - 2f(x)$.

We can express f in the form

$$f = f_1 - f_2$$

where f_1 and f_2 are continuous and increasing on $[-2\pi, 2\pi]$. Clearly f_1 and f_2 are uniformly continuous on $[-2\pi, 2\pi]$.

Choose any $\varepsilon > 0$. We can find a positive number δ_0 with $0 < \delta_0 < \dfrac{\pi}{2}$ such that

$$|f_i(x') - f_i(x'')| < \frac{\varepsilon}{20}.\ (i=1,2)$$

for all x', x'' in $[-2\pi, 2\pi]$ with $|x'-x''|<\delta_0$.

Take $a = \frac{1}{4}\delta_0$, and any $x \in [-\pi, \pi]$ and any positive integer n.

We have

$$S_n(f;\, x) - f(x) = \frac{1}{\pi}\int_0^a \phi(x,t).\frac{\sin(2nt1)t}{\sin t}dt$$

$$+ \frac{1}{\pi}\int_a^{\frac{\pi}{2}} \phi(x,t).\frac{\sin(2n+1)t}{\sin t}dt$$

$= I_n(x) + J_n(x)$ (say). ... ... (9.20)

Now $I_n(x) = I_n^1(x) - I_n^2(x)$, where

$$I_n^i(x) = \frac{1}{\pi}\int_0^a \{f_i(x+2t) - f_i(x)\}\frac{\sin(2n+1)t}{\sin t}dt$$

$$+\frac{1}{\pi}\int_0^a \{f_i(x-2t) - f_i(x)\}.\frac{\sin(2n+1)t}{\sin t}dt$$

$$= \frac{1}{\pi}\{f_i(x+2a) - f_i(x)\}\int_\xi^a \frac{\sin(2n+1)t}{\sin t}dt$$

$$+ \frac{1}{\pi}\{f_i(x-2a) - f_i(x)\}\int_\eta^a \frac{\sin(2m+1)t}{\sin t}dt$$

[Using second Mean value Theorem)

where $0 \le \xi, \eta \le a$.

So $$\left|I_n^i(x)\right| \le \frac{2}{\pi}.\frac{\varepsilon}{20}\left(\frac{\pi}{2} + 4\sqrt{\pi}\right) < \frac{\varepsilon}{\pi}.$$

Hence $$\left|I_n(x)\right| < \frac{2\varepsilon}{\pi}. < \varepsilon \qquad \text{... ... (9.21)}$$

Now consider $J_n(x)$.

Let $k = max\ \{|f(x)| : -\pi \le x \le \pi\}$.

and $M = \frac{1+4k}{\sin^2 a}$.

Then $|\phi(x,t)| = |f(x+2t) + f(x-2t) - 2f(x)| \leq 4k$.

Since $f(x)$ and $\sin x$ are uniformly continuous on $[-2\pi, 2\pi]$ we can find $a\ \delta \left(0<\delta<\dfrac{\pi}{2}\right)$ such that

$$|f(x') - f(x'')| < \frac{\varepsilon}{2M} \text{ and } |\sin x' - \sin x''| < \frac{\varepsilon}{2M}$$

for all x', x'' in $[-2\pi, 2\pi]$ with $|x'-x''| < \delta$.

Let $a = t_0 < t_1 < t_2 < \cdots t_m = \dfrac{\pi}{2}$ be a subdivision of $\left[a, \dfrac{\pi}{2}\right]$ with

$max\ (t_i - t_{i-1}) < \dfrac{1}{2}\delta$.

We have

$$J_n(x) = \frac{1}{\pi}\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}} \psi(x,t)\ \sin(2n+1)\,t\ dt,$$

where $\psi(x, t) = \phi(x,t)/\sin t$.

$$= \frac{1}{\pi}\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}} \{\psi(x,t) - \psi(x,t_i)\}\sin(2n+1)\,t\ dt$$

$$+ \frac{1}{\pi}\sum_{i=0}^{m-1}\psi(x,t_i)\int_{b_i}^{b_{i+1}} \sin(2n+1)t\ dt.$$

$= J_n'(x) + J_n''(x)$ (say).

Now $|\psi(x,t)| = \left|\dfrac{\phi(x,t)}{\sin t}\right| \leq \dfrac{4k}{\sin^2 a}.\left(a \leq t \leq \dfrac{\pi}{2}\right)$

For $t_i \leq t \leq t_{i+1}$

$$|\phi(x,t) - \phi(x,t_i)|$$

$$\leq |f(x+2t) - f(x+2t_i)| + |f(x-2t) - f(x-2t_i)|$$

$$< \frac{\varepsilon}{M}.$$

and $|\psi(x,t) - \psi(x,t_i)| = \left|\dfrac{\phi(x,t)}{\sin t} - \dfrac{\phi(x,t_i)}{\sin t_i}\right|$

$\leq \text{cosec}^2\, a\ \{ |\phi(x,t) - \phi(x,t_i)| + |\phi(x,t_i)|\ |\sin t - \sin t_i| \}$

$$\leq \text{cosec}^2 a \left\{\frac{\varepsilon}{M} + 4k.\frac{\varepsilon}{2M}\right\} = (1+2k)\frac{\varepsilon}{M \sin^2 a}$$

$$= \left(\frac{1+2k}{1+4k}\right). \varepsilon < \varepsilon.$$

So $$|J_n'(x)| \leq \frac{\varepsilon}{\pi}.\sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} dt = \frac{\varepsilon}{\pi}\left(\frac{\pi}{2} - 0\right) < \frac{1}{2}\varepsilon.$$

and $$|J_n''(x)| \leq \frac{8km}{\pi \sin^2 a} \cdot \frac{1}{2n+1} \to 0 \text{ as } n \to \infty.$$

So we can find a positive integer n_0 such that

$$|J_{n(x)}''| < \frac{1}{2}\varepsilon \text{ when } n \geq n_0.$$

Hence for $n \geq n_0$

$$|J_n(x)| < \varepsilon \qquad \ldots \qquad \ldots \qquad (9.22)$$

From (9.20), (9.21) and (9.22) we obtain

$$|S_n(f;x) - f(x)| < 2\varepsilon$$

for all x in $[-\pi, \pi]$ when $n \geq n_0$.

Therefore $\{S_n(f;x)\}$ converges uniformly to $f(x)$ on $[-\pi, \pi]$.

Theorem 9.15. Let the functions f and g be summable on $[-\pi,\pi]$ and be periodic with period 2π. Let

$$f(x) \sim \frac{1}{2}a_0 + \sum_{v=1}^{\infty} (a_v \cos vx + bv \sin vx),$$

$$g(x) \sim \frac{1}{2}\alpha_0 + \sum_{v=1}^{\infty} (\alpha_v \cos vx + b_v \sin vx).$$

If one of f and g is BV on $[-\pi, \pi]$. then

$$\frac{1}{2}a_0\,\alpha_0 + \sum_{v=1}^{\infty} (a_v\alpha_v + b_v\,\beta_v) = \frac{1}{\pi}\int_{-\pi}^{\pi} fg\,dx.$$

Proof : Suppose that f is BV on $[-\pi,\pi]$. Then f is continuous on $[-\pi, \pi]$ except a countable set. By Jordan Test $S_n(f;x) \to f(x)$ $a.\ e.$ on $[-\pi, \pi]$. We now show that the sequence $\{S_n(f;x)\}$ is uniformly bounded on $[-\pi, \pi]$. Let x_0 be any point in $[-\pi, \pi]$. Then

$$S_n(f\,;x_0) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \{f(x_0+2t) + f(x_0-2t)\}\,\frac{\sin(2n+1)t}{\sin t}\,dt$$

$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \phi(t).\frac{\sin(2n+1)t}{\sin t}\,dt,$$

where $\phi(t) = f(x_0+2t) + f(x_0-2t)$.

Clearly ϕ is BV on $\left[0, \frac{\pi}{2}\right]$. We can express ϕ in the from $\phi = \phi_1 - \phi_2$, where ϕ_1 and ϕ_2 are non-negative and increasing on $\left[0, \frac{\pi}{2}\right]$ and $\phi_1(0) = 0$, $\phi_2(0) = 0$. So

$$S_n(f\,;x_0) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \phi_1(t)\frac{\sin(2n+1)t}{\sin t}\,dt - \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \phi_2(t)\frac{\sin(2n+1)t}{\sin t}\,dt$$

$$= \frac{1}{\pi}\phi_1\left(\frac{\pi}{2}\right)\int_{\xi}^{\frac{\pi}{2}} \frac{\sin(2n+1)t}{\sin t}\,dt - \frac{\phi_2\left(\frac{\pi}{2}\right)}{\pi}\int_{\eta}^{\frac{\pi}{2}} \frac{\sin(2n+1)t}{\sin t}\,dt$$

where $0 \le \xi, \eta \le \frac{\pi}{2}$.

Using Lemma 9.2 we get

$$|S_n(f\,;x_0)| \le \frac{4k}{\pi}\left(\frac{\pi}{2}+4\sqrt{\pi}\right), \qquad \dots \quad \dots \quad (9.23)$$

where $k = \text{Sup}\,\{|f(x)| : -\pi \le x \le \pi\}$.

Since x_0 is *an* arbitrary point of $[-\pi, \pi]$ it follows from (9.23) that $\{S_n(f\,;x)\}$ is uniformly bounded on $[-\pi, \pi]$.

The function g is summable on $[-\pi, \pi]$. So by Lebesgue doninated convergence Theorem we have

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} g(x)\,S_n(f\,;x) = \int_{-\pi}^{\pi} f(x)g(x)\,dx \qquad \dots \quad \dots \quad (9.24)$$

Now

$$\int_{-\pi}^{\pi} g(x) S_n (f;\ x)dx$$

$$= \frac{1}{2}a_0\int_{-\pi}^{\pi} g(x)dx + \sum_{\nu=1}^{n}\left(a_\nu \int_{-\pi}^{\pi} g(x)\cos \nu x dx + b_\nu \int_{-\pi}^{\pi} g(x)\sin \nu x dx\right)$$

$$= \pi\left\{\frac{1}{2}a_0\alpha_0 + \sum_{\nu=1}^{n}(a_\nu\alpha_\nu + b_\nu\beta_\nu)\right\}.$$

Hence from (9.24) we obtain

$$\frac{1}{2}a_0\alpha_0 + \sum_{\nu=1}^{\infty}(a_\nu\alpha_\nu + b_\nu\beta_\nu) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx.$$

Corollary 9.15.1. Let f be summable and periodic with period 2π. If u be any point in $[-\pi, \pi]$, then

$$\int_0^u f(x)dx = \frac{1}{2}a_0 u + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b\nu \sin \nu x)\ dx.$$

Proof : Let $0 < u \leq \pi$. Define the function g on as follows.

$g(x) = 1$ for all x in $(0, u)$

$= 0$ else where in $[-\pi, \pi]$

Further $g(x + 2\pi) = g(x)$ for all real x.

Then clearly g is BV on $[-\pi, \pi]$. So by

Theorem 9.15 we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx = \frac{1}{2}a_0\alpha_0 + \sum_{\nu=1}^{\infty} (a_\nu\ \alpha_\nu + b_\nu\ \beta_\nu) \quad \ldots \quad \ldots \quad (9.25)$$

where $$f(x) \sim \frac{1}{2}a_\nu + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

$$g(x) \sim \frac{1}{2}\alpha_0 + \sum_{\nu=1}^{\infty} (\alpha_\nu \cos \nu x + \beta_\nu \sin \nu x).$$

Now $$\int_{-\pi}^{\pi} f(x)g(x)dx = \int_0^u f(x)dx$$

and

$$a_\nu \alpha_\nu + b_\nu \beta_\nu$$

$$= \frac{1}{\pi}\left[a_\nu \int_{-\pi}^{\pi} g(x)\cos \nu x dx + b_\nu \int_{-\pi}^{\pi} g(x)\sin \nu x dx\right]$$

$$= \frac{1}{\pi}\int_0^u (a_\nu \cos \nu x + b_\nu \sin \nu x)dx.$$

Hence substituting in (9.25) we obtain

$$\int_0^u f(x)dx = \frac{1}{2}a_0 u + \sum_{\nu=1}^{\infty}\int_0^u (a_\nu \cos \nu x + b_\nu \sin \nu x)\, dx. \quad \ldots \quad \ldots \quad (9.26)$$

If $-\pi \le u < 0$, define g by $g(x) = 1$ in $(u, 0)$ and $g(x) = 0$ in $[-\pi, \pi] \setminus (u,0)$ and $g(x+2\pi) = g(x)$ for all real x.

Proceeding as above we see that (9.26) holds.

Corollary 9.15.2. Let f be BV on $[-\pi, \pi]$ and periodic with period 2π. Then

$$\frac{1}{2}a_0^2 + \sum_{\nu=1}^{\infty}\left(a_\nu^2 + b_\nu^2\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx,$$

where $f(x) \sim \frac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x)$.

The result follows from Theorem 9.15 by taking $g = f$.

Theorem 9.16. Let the function f be square summable on $[-\pi,\pi]$ and periodic with period 2π and let

$$f(x) \sim \frac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b\nu \sin \nu x).$$

Then

$$\frac{1}{2}a_0^2 + \sum_{\nu=1}^{\infty}\left(a_\nu^2 + b_\nu^2\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx.$$

Proof : Choose any $\varepsilon > 0$. Then there is a function g absolutely continuous on $[-\pi, \pi]$ such that

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|f - g|^2\, dx < \varepsilon.$$

Define g outside $[-\pi, \pi]$ by periodicity with period 2π. Let

$$g(x) \sim \frac{1}{2}\alpha_0 + \sum_{\nu=1}^{\infty} (\alpha_\nu \cos \nu x + \beta_\nu \sin \nu x).$$

Then $f(x)-g(x) \sim \frac{1}{2}(a_0-\alpha_0)+\sum_{\nu=1}^{\infty} [(a_\nu-\alpha_\nu) \cos\nu x+(b_\nu-\beta_\nu) \sin\nu x]$

By Bessel's inequality we have

$$\frac{1}{2}(a_0-\alpha_0)^2+\sum_{\nu=1}^{\infty}\left[(a_\nu-\alpha_\nu)^2+(b_\nu-\beta_\nu)^2\right] \le \frac{1}{\pi}\int_{-\pi}^{\pi}|f-g|^2\,dx < \varepsilon. \quad \ldots \quad (9.27)$$

Since g is BV on $[-\pi, \pi]$,

$$\frac{1}{2}\alpha_0^2+\sum_{\nu=1}^{\infty}\left(\alpha_\nu^2+\beta_\nu^2\right)=\frac{1}{\pi}\int_{-\pi}^{\pi} g^2\,dx.$$

$$\frac{1}{2}\alpha_0(a_0-\alpha_0)+\sum_{\nu=1}^{\infty}\left[\alpha_\nu(a_\nu-\alpha_\nu)+\beta_\nu(b_\nu-\beta_\nu)\right]=\frac{1}{\pi}\int_{-\pi}^{\pi} g(f-g)\,dx.$$

Choose positive integer N such that for $n \ge N$,

$$\left|\frac{1}{\pi}\int_{-\pi}^{\pi} g^2\,dx-\left[\frac{1}{2}\alpha_0^2+\sum_{\nu=1}^{n}\left(\alpha_\nu^2+\beta_\nu^2\right)\right]\right|<\varepsilon \quad \ldots \quad \ldots \quad (9.28)$$

$$\left|\frac{1}{\pi}\int_{-\pi}^{\pi} g(f-g)\,dx-\left[\frac{1}{2}\alpha_0(a_0-\alpha_0)+\sum_{\nu=1}^{n}\left\{\alpha_\nu(a_\nu-\alpha_\nu)+\beta_\nu(b_\nu-\beta_\nu)\right\}\right]\right|<\varepsilon \quad \ldots \quad \ldots (9.29)$$

Let $S_n = \frac{1}{2}\alpha_0^2+\sum_{\nu=1}^{n}\left(\alpha_\nu^2+\beta_\nu^2\right)$,

$$T_n = \frac{1}{2}\alpha_0\,(a_0-\alpha)+\sum_{\nu=1}^{n}\{\alpha_\nu\,(a_\nu-\alpha_\nu)+\beta_\nu\,(b_\nu-\beta_\nu)\},$$

$$R_n = \frac{1}{2}(a_0-\alpha_0)^2+\sum_{\nu=1}^{n}\{(a_\nu-\alpha_\nu)^2+(b_\nu-\beta_\nu)^2\}.$$

Since $\quad a_0^2 = \left[(a_0 - \alpha_0) + \alpha_0\right]^2 = \alpha_0^2 + 2\alpha_0(a_0 - \alpha_0) + (a_0 - \alpha_0)^2$

$$a_\nu^2 = \left[\alpha_\nu + (a_\nu - \alpha_\nu)\right]^2 = \alpha_\nu^2 + 2\alpha_\nu(a_\nu - \alpha_\nu) + (a_\nu - \alpha_\nu)^2$$

$$b_\nu^2 = \left[\beta_\nu + (b_\nu - \beta_\nu)\right]^2 = \beta_\nu^2 + 2\beta_\nu(b_\nu - \beta_\nu) + (b_\nu - \beta_\nu)^2$$

we have

$$\frac{1}{2}a_0^2 + \sum_{\nu=1}^{n}\left(a_\nu^2 + b_\nu^2\right) = S_n + 2\,T_n + R_n\,.$$

Take any $n \geq N$. Then using (9.27), (9.28) and (9.29) we get

$$\left|\frac{1}{\pi}\int_{-\pi}^{\pi} f^2 dx - \left\{\frac{1}{2}a_0^2 + \sum_{\nu=1}^{n}\left(a_\nu^2 + b_\nu^2\right)\right\}\right|$$

$$= \left|\frac{1}{\pi}\int_{-\pi}^{\pi}\{g + (f - g)\}^2 dx - (S_n + 2\,T_n + R_n)\right|$$

$$\leq \left|\frac{1}{\pi}\int_{-\pi}^{\pi} g^2 dx - S_n\right| + 2\left|\frac{1}{\pi}\int_{-\pi}^{\pi} g(f-g)dx - T_n\right| + \left|\frac{1}{\pi}\int_{-\pi}^{\pi}(f-g)^2 dx - R_n\right|$$

$< 5\,\varepsilon$.

This gives that

$$\frac{1}{2}a_0^2 + \sum_{\nu=1}^{\infty}\left(a_\nu^2 + b_\nu^2\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2 dx.$$

Note : This result is known as Parseval's Indentity.

Theorem 9.17. (Riesz–Fischer Theorem).

Let $a_0, a_1, a_2, \ldots; b_1, b_2, b_3, \ldots$ be two sequences of real members such that

$$\frac{1}{2}a_0^2 + \sum_{\nu=1}^{\infty}\left(a_\nu^2 + b_\nu^2\right) < +\infty.$$

Then there exists *a* function f square summable on $[-\pi, \pi]$ such that

$$f(x) \sim \frac{1}{2}a_0 + \sum_{\nu=1}^{\infty}(a_\nu \cos x + b\nu \sin \nu x)$$

and $$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2 dx = \frac{1}{2}a_0^2 + \sum_{\nu=1}^{\infty}\left(a_\nu^2 + b_\nu^2\right).$$

Proof: Let $S = \frac{1}{2}a_0^2 + \sum_{\nu=1}^{\infty}\left(a_\nu^2 + b_\nu^2\right).$

Choose the sequence $\{n_r\}$ ($n_1 < n_2 < n_3 < \ldots.$) of positive integers such that

$$\sum_{\nu=n_r}^{n_{r+1}}{}' \left(a_\nu^2 + b_\nu^2\right) < \frac{S}{2^{2r}} \ (r = 1,2,3,\ldots).$$

Let $f_0(x) = \frac{1}{2}a_0 + \sum_{\nu=1}^{n_1} (a_\nu \cos \nu x + b\nu \sin \nu x),$

$$f_r(x) = \sum_{\nu=n_r+1}^{n_{r+1}} (a_\nu \cos \nu x + b\nu \sin \nu x) \ (r = 1, 2, 3, \ldots\ldots..)$$

We have

$$\int_{-\pi}^{\pi} f_0^2(x)dx = \pi\left[\frac{1}{2}a_0^2 + \sum_{\nu=1}^{n_1}\left(a_\nu^2 + b_\nu^2\right)\right] \leq \pi s$$

and

$$\int_{-\pi}^{\pi} f_r^2(x)dx = \pi\left[\sum_{\nu=n_r+1}^{n_{r+1}}{}' \left(a_\nu^2 + b_\nu^2\right)\right] \leq \frac{\pi s}{2^{2r}}.$$

By Cauchy's inequality we get

$$\int_{-\pi}^{\pi} |f_r(x)|dx \leq \sqrt{2\pi}\left\{\int_{-\pi}^{\pi} f_r^2(x)dx\right\}^{\frac{1}{2}} \leq \sqrt{2\pi}\cdot\frac{\sqrt{\pi s}}{2^r}$$

$(r = 0, 1, 2, 3,\ldots)$

This gives that

$$\sum_{r=0}^{\infty}\int_{-\pi}^{\pi} |f_r(x)|dx \leq \pi\sqrt{2s}\sum_{r=0}^{\infty}\frac{1}{2^r} < +\infty.$$

Therefore the series $\sum_{r=0}^{\infty} f_r(x)$ converges absolutely *a. e.* to *a* function *f(x)* which is summable on $[-\pi, \pi]$. (See Th. 7.7)

Let r be any positive integer. Then for any integer $m \geq 0$,

$$\int_{-\pi}^{\pi} |f_m(x) f_r(x)| dx \leq \left\{\int_{-\pi}^{\pi} f_m^2(x) dx\right\}^{\frac{1}{2}} \left\{\int_{-\pi}^{\pi} f_r^2(x) dx\right\}^{\frac{1}{2}}$$

$$\leq \left(\frac{\pi s}{4^m} \cdot \frac{\pi s}{4^r}\right)^{\frac{1}{2}} = \frac{\pi s}{2^{m+r}}.$$

Since $f(x) f_r(x)$ is measurable and

$$|f(x) f_r(x)| \leq \sum_{m=0}^{\infty} |f_m(x)\, f_r(x)|,$$

$$\int_{-\pi}^{\pi} |f(x) f_r(x)| dx \leq \sum_{m=0}^{\infty} \int_{-\pi}^{\pi} |f_m(x) f_r(x)| dx$$

$$\leq \frac{\pi s}{2^r} \sum_{m=0}^{\infty} \frac{1}{2^m} = \frac{2\pi s}{2^r}.$$

So $\int_{-\pi}^{\pi} f^2(x) dx \leq \sum_{r=0}^{\infty} \int_{-\pi}^{\pi} |f(x) f_r(x)| dx \leq 4\pi s < +\infty$

Hence $f(x)$ is square summable on $[-\pi, \pi]$. Let n be any non-negative integer. We have

$$\int_{-\pi}^{\pi} f^2(x) \cos nx\, dx = \sum_{r=0}^{\infty} \int_{-\pi}^{\pi} f_r(x) \cos nx dx = \pi a_n$$

$$\int_{-\pi}^{\pi} f(x) \sin nx dx = \sum_{r=0}^{\infty} \int_{-\pi}^{\pi} f_r(x) \sin nx dx = \pi b_n$$

Thus $a_0, a_1, a_2, \ldots$; $b_1, b_2, b_3, \ldots$ are Fourier coefficients of f on $[-\pi. \pi]$.

$$\int_{-\pi}^{\pi} f^2(x) dx = \sum_{r=0}^{\infty} \int_{-\pi}^{\pi} f(x) f_r(x) dx$$

$$= \pi\left[\frac{1}{2} a_0^2 + \sum_{\nu=0}^{\infty} \left(a_\nu^2 + b_\nu^2\right)\right] = \pi s.$$

This proves the theorem.

Theorem 9.18. Let the functions f and g be summable on $[-\pi, \pi]$ and periodic with period 2π. If they have the same Fourier series on $[-\pi,\pi]$, then $f(x) = g(x)$ *a.e.* on $[-\pi,\pi]$.

Proof : By Theorem 9.13, we have for $-\pi \le x \le \pi$,

$$\int_0^x f(x)dt = \frac{1}{2}a_0 x + \sum_{\nu=1}^{\infty}\int_0^x (a\nu \cos\nu t + b_\nu \sin \nu t)dt$$

and $$\int_0^x g(t)dt = \frac{1}{2}a_0 x + \sum_{\nu=1}^{\infty}\int_0^x (a_\nu \cos\nu t + b_\nu \sin\nu t)\, dt.$$

So $\phi(x) = \int_0^x [f(f)-g(t)]dt = 0$ for $-\pi \le x \le \pi$.

Then $\phi(x)$ is absolutely continuous on $[-\pi, \pi]$ and $\phi'(x) = 0$ *a.e.* on $[-\pi, \pi]$. Since $\phi'(x) = f(x) - g(x)$ *a.e.* on $[-\pi, \pi]$, $f(x) = g(x)$ *a.e.* on $[-\pi, \pi]$.

Cesaro or (C–1) summability of Fourier series.

Let $\{s_n\}$ be *a* sequence of real numbers. Let

$$\sigma_n = \frac{1}{n}(s_1 + s_2 + \ldots + s_n)\ (n=1,2,3,\ldots\ldots).$$

If the sequence $\{\sigma_n\}$ converges to l(say) we say that the sequence $\{s_n\}$ is $(c, 1)$summable to l. If the sequence $\{s_n\}$coverges to l, then it is easy to see that $\sigma_n \to l$ as $n \to \infty$ and so $\{s_n\}$ is $(c,1)$ summable to l. But the converse is not true. This can be shown by an example.

Take $s_n = (-1)^{n-1} (n = 1,2,3,\ldots)$. Then $\{s_n\}$ is an oscillatory sequence. We have $\sigma_n = \frac{1}{n}$ if n is odd and $\sigma_n = 0$ when n is even. This gives that $\sigma_n \to 0$ as $n \to \infty$ and so $\{s_n\}$ is $(C,1)$ summable to 0 (zero).

An infinite series Σu_n is said to be $(c,1)$ summable to l (say) if the sequence $\{s_n\}$ is $(c,1)$ summable to l, where $s_n = u_1 + u_2 + \cdots + u_n$ $(n = 1, 2, 3,\ldots)$.

Theorem 9.19. (Fejer's Integral).

Let the function f be summable on $[-\pi, \pi]$ and periodic with period 2π and

$$f(x) \sim \frac{1}{2} a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x).$$

For $x \in [-\pi, \pi]$ and positive integer n,

$$\sigma_n (f; x) = \frac{1}{\pi n} \int_0^{\frac{\pi}{2}} [f(x+2t) + f(x-2t)].\left(\frac{\sin nt}{\sin t}\right)^2 dt,$$

where

$$\sigma_n(f; x) = \frac{1}{n} [S_0(f, x) + S_1(f; x) + S_1(f; x) + S_2(f; x) + \dots + S_{n-1}(f; x)],$$

$$S_0(f; x) = \frac{1}{2} a_0 \text{ and } S_n(f; x) = \frac{1}{2} a_0 + \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x).$$

Proof : By Dirichlet's Integral we have

$$S_n(f; x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} [f(x+2t) + f(x-2t)] \frac{\sin(2n+1)t}{\sin t} dt.$$

So $\quad S_0(f; x) + S_1(f; x) + S_2(f; x) + \dots + S_{n-1}(f; x)$

$$= \frac{1}{\pi} \int_0^{\frac{\pi}{2}} [f(x+2t) + f(x-2t)]. \sum_{\nu=0}^{n-1} \frac{\sin(2\nu+1)t}{\sin t} dt$$

Now $\quad 2 \sin t \sin (2\nu + 1)t = \cos 2\nu t - \cos 2(\nu + 1)t$

and $2 \sin t. \sum_{\nu=0}^{n-1} \sin (2\nu + 1)t = \sum_{\nu=0}^{n-1} [\cos 2\nu t - \cos 2(\nu + 1)t]$

$$= 1 - \cos 2nt = 2 \sin^2 nt.$$

Therefore

$$\sigma_n(f; x) = \frac{1}{n\pi} \int_0^{\frac{\pi}{2}} [f(x+2t) + f(x-2t)]\left(\frac{\sin nt}{\sin t}\right)^2 dt.$$

Theorem 9.20. Let the function f be summable on $[-\pi, \pi]$ and periodic with period 2π. Then *a* necessary and sufficient condition for $(C,1)$ summability of the Fourier series of f at $x_0 \in [-\pi, \pi]$ to $S(x_0)$ is that

$$\lim_{n\to\infty} \frac{1}{n\pi} \int_0^{\delta} \left[f(x_0+2t)+f(x_0-2t)-2s(x_0) \right] \left(\frac{\sin nt}{\sin t} \right) dt = 0,$$

where δ is any positive number with $0 < \delta < \frac{\pi}{2}$.

Proof : Let $x_0 \in [-\pi, \pi]$ and n be any positive integer. By Fejer's Integeral we have

$$\sigma_n(f; x_0) = \frac{1}{n\pi} \int_0^{\frac{\pi}{2}} [(x_0+2t)+f(x_0-2t)] \left(\frac{\sin nt}{\sin t} \right)^2 dt.$$

By actual evaluation we get

$$\int_0^{\frac{\pi}{2}} \left(\frac{\sin nt}{\sin t} \right)^2 dt = \frac{n\pi}{2}.$$

Write $\phi(t) = f(x_0+2t) + f(x_0-2t) - 2\, s(x_0)$.

Then $\phi(t)$ is summable on any finite interval. We have

$$\sigma_n(f; x_0) - S(x_0) = \frac{1}{n\pi} \int_0^{\frac{\pi}{2}} [f(x_0+2t) + f(x_0-2t) - 2s(x_0)] \left(\frac{\sin nt}{\sin t} \right)^2 dt.$$

$$= \frac{1}{n\pi} \int_0^{\delta} \phi(t) \left(\frac{\sin nt}{\sin t} \right)^2 dt + \frac{1}{n\pi} \int_{\delta}^{\frac{\pi}{2}} \phi(t) \left(\frac{\sin nt}{\sin t} \right) dt$$

(wher $0 < \delta < \frac{\pi}{2}$)

$$= I_n + J_n \text{ (say)} \qquad \dots \qquad \dots \qquad (9.29)$$

Now

$$|J_n| \le \frac{1}{n\pi} \int_{\delta}^{\frac{\pi}{2}} \frac{|\phi(t)|}{\sin^2 t} dt \to 0 \text{ as } n \to \infty.$$

that is,

$$\lim_{n\to\infty} J_n = 0. \qquad \dots \qquad \dots \qquad (9.30)$$

Suppose that $I_n \to 0$ as $n \to \infty$ 0 as $n \to \infty$, that is,

$$\lim_{n\to\infty} \frac{1}{n\pi}\int_0^{\delta}\phi(t)\left(\frac{\sin nt}{\sin t}\right)^2 dt = 0. \tag{9.31}$$

Then from (9.29), (9.30) and (9.31) we see that $\sigma_n(f;x_0) \to 0$ as $n \to \infty$, that is, the Fourier series of f at x_0 is $(c,1)$ summable to $s(x_0)$. Conversely suppose that $\sigma_n(f;x_0) \to s(x_0)$ as $n \to \infty$. Then from (9.29) and (9.30) we see that $I_n \to 0$ as $n \to \infty$, that is, (9.31) holds.

Theorem 9.21. (Fejer's Theorem). Let the function f be summable on $[-\pi, \pi]$ and be periodic with period 2π. If at the point $x_0 \in [-\pi, \pi]$, $f(x_0+0)$ and $f(x_0-0)$ exist finitely, then the Fourier series of f at x_0 is $(c, 1)$ summable to

$$\frac{1}{2}[f(x_0+0)+f(x_0-0)]$$

Proof : let $x_0 \in [-\pi, \pi]$. Suppose that $f(x_0+0)$ and $f(x_0-0)$ exist finitely. Let n be any positive integer. By Fejer's Integral we have

$$\sigma_n(f;x_0) = \frac{1}{n\pi}\int_0^{\frac{\pi}{2}} [f(x_0+2t)+f(x_0-2t)].\left(\frac{\sin nt}{\sin t}\right)^2 dt.$$

Write $s(x_0) = \dfrac{1}{2}[f(x_0+0) - f(x_0-0)]$

and $\qquad \phi(t) = f(x_0+2t) + f(x_0-2t) - 2\, s(x_0).$

Then ϕ is summaable on any finite interval and $\phi(t) \to 0$ as $t \to 0+$.

Choose any $\varepsilon > 0$. We can find a δ $(0 < \delta < \frac{\pi}{2})$ such that

$$|\phi(t)| < \varepsilon \text{ for } 0 \le t \le \delta.$$

Now

$$\sigma_n(f;x_0) - s(x_0) = \frac{1}{n\pi}\int_0^{\delta}\phi(t)\left(\frac{\sin nt}{\sin t}\right)^2 dt + \frac{1}{n\pi}\int_{\delta}^{\frac{\pi}{2}}\phi(t)\left(\frac{\sin nt}{\sin t}\right)^2 dt$$

$$= I_n + J_n \text{ (say).}$$

$$|I_n| \le \frac{1}{n\pi}\int_0^{\delta}|\phi(t)|.\left(\frac{\sin nt}{\sin t}\right)^2 dt$$

$$\leq \frac{\varepsilon}{n\pi}\int_0^{\delta}\left(\frac{\sin nt}{\sin t}\right)^2 dt \leq \frac{\varepsilon}{n\pi}\int_0^{\frac{\pi}{2}}\left(\frac{\sin nt}{\sin t}\right)^2 dt$$

$$< \varepsilon$$

and $|J_n| \leq \dfrac{1}{n\pi}\displaystyle\int_{\delta}^{\frac{\pi}{2}} \frac{|\phi(t)|}{\sin^2 t}\, dt \to 0$ as $n \to \infty$.

Hence we can determine a positive integer n_0 such that

$$|J_n| < \varepsilon \text{ when } n \geq n_0.$$

So $\qquad |\sigma_n(f; x_0) - s(x_0)| < 2\varepsilon$ when $n \geq n_0$.

Therefore $\displaystyle\lim_{n\to\infty} \sigma_n(f; x_0) = s(x_0)$.

This completes the proof.

Corollary 9.21.1. Let the function f be summable on $[-\pi, \pi]$, and be periodic with period 2π. If at $x_0 \in [-\pi,\pi]$, $f(x_0+0)$ and $f(x_0-0)$ exist finitely and the Fourier series at x_0 converges, then the sum of the series is

$$\frac{1}{2}[f(x_0 + 0) + f(x_0 - 0)].$$

Theorem 9. 22. (Lebesgue). Let the function f be summable on the interval $[-\pi, \pi]$ and periodic with period 2π. If $x_0 \in (-\pi, \pi)$ is a Lebesgue point of f, then the Fourier series of f at x_0 is $(C, 1)$ summable to $f(x_0)$.

Proof : Let $f(x) \sim \dfrac{1}{2} a_0 + \displaystyle\sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x)$,

$$S_0(f; x) = \frac{1}{2} a_0,\ S_n(f; x) = \frac{1}{2} a_0 + \sum_{\nu=1}^{n-1} (a_\nu \cos\nu x + b_\nu \sin\nu x)$$

and $\sigma_n(f; x) = \dfrac{1}{2}[s_0(f; x) + S_1(f; x) + S_2(f; x) + \cdots + S_{n-1}(f.x)]$.

Suppose that $x_0 \in (-\pi, \pi)$ is a Lebesgue point of f. By Fejer's Integeral

$$\sigma_n(f; x_0) - f(x_0) = \frac{1}{n\pi}\int_0^{\frac{\pi}{2}} \varphi(t).\left(\frac{\sin nt}{\sin t}\right)^2 dt, \qquad \ldots \qquad (9.32)$$

where $\phi(t) = f(x_0 + 2t) + f(x_0 - 2t) - 2f(x_0)$.

Let $\psi(t) = \int_0^t |\phi(u)|\, du$. We have for $h\ (\neq 0)$,

$$\left|\frac{\psi(h)}{h}\right| \leq \left|\frac{1}{h}\int_0^h \left|f(x_0+2t)-f(x_0)\right| dt\right| + \left|\frac{1}{h}\int_0^h \left|f(x_0-2t)-f(x_0)\right| dt\right|$$

$$\leq \left|\frac{1}{2h}\int_{x_0}^{x_0+2h} |f(t)-f(x_0)|\,dt\right| + \left|\frac{1}{2h}\int_{x_0}^{x_0-2h} |f(t)-f(x_0)|\,dt\right|$$

$\to 0$ as $h \to 0$ [$\because x_0$ is *a* Lebesgue point of f].

Choose any $\varepsilon > 0$. We determine α $(0 < \alpha < \frac{\pi}{2})$ such that

$$\left|\frac{1}{t}\psi(t)\right| < \varepsilon \text{ for } 0 < |t| \leq \alpha. \qquad \ldots \qquad (9.33)$$

Take any t in $(\alpha, \frac{\pi}{2})$.

Since $\quad 1 < \dfrac{t}{\sin t} < \dfrac{\pi}{2}$, we have

$$\frac{1}{n}\left(\frac{\sin nt}{\sin t}\right)^2 \leq \frac{1}{n}\cdot\frac{1}{\sin^2 t} \leq \frac{1}{n}\cdot\frac{\pi^2}{4t^2}$$

$$< \frac{1}{n}\cdot\frac{\pi^2}{4\alpha^2} \to 0 \text{ as } n \to \infty.$$

Determine positive integer n_0 $(> \frac{1}{\alpha})$ such that

$$\frac{1}{n}\cdot\left(\frac{\sin nt}{\sin t}\right)^2 < \varepsilon \text{ when } n \geq n_0.$$

If $n \geq n_0$, then $\frac{1}{n} < \alpha$; so from (9.33)

$$n\,\psi\left(\frac{1}{n}\right) < \varepsilon \qquad \ldots \qquad \ldots \qquad (9.34)$$

Take any $n \geq n_0$. We have from (9.32)

$$\sigma_n(f;x_0) - f(x_0) = \frac{1}{n\pi}\int_0^{\frac{1}{n}} \phi(t)\left(\frac{\sin nt}{\sin t}\right)^2 dt$$

$$+\frac{1}{n\pi}\int_{\frac{1}{n}}^{\alpha} \phi(t).\left(\frac{\sin nt}{\sin t}\right)^2 dt + \frac{1}{n\pi}\int_{\alpha}^{\frac{\pi}{2}} \phi(t).\left(\frac{\sin nt}{\sin t}\right)^2 dt$$

$$= I_1(n) + I_2(n) + I_3(n) \text{ (say)} \qquad \dots \qquad \dots \qquad (9.35)$$

$$|I_3(n)| \leq \frac{\operatorname{cosec}^2\alpha}{n\pi}\int_{\alpha}^{\frac{\pi}{2}} |\phi(t)|\alpha t \to 0 \text{ as } n \to \infty.$$

Determine positive integer $n_1(\geq n_0)$ such that

$$|I_3(n)| < \varepsilon \text{ when } n \geq n_1 \qquad \dots \qquad \dots \qquad (9.36)$$

$$|I_1(n)| \leq \frac{1}{n\pi}\int_0^{\frac{1}{n}} |\phi(t)|\left(\frac{\sin nt}{\sin t}\right)^2 dt$$

$$\leq \frac{1}{n\pi}\int_0^{\frac{1}{n}} |\phi(t)|\frac{n^2\pi^2}{4} dt$$

$$= \frac{\pi}{4}n\psi\left(\frac{1}{n}\right) < \frac{\pi}{4}\varepsilon < \varepsilon \qquad \dots \qquad \dots \qquad (9.37)$$

$$|I_2(n)| \leq \frac{1}{n\pi}\int_{\frac{1}{n}}^{\alpha} |\phi(t)|\left(\frac{\sin nt}{\sin t}\right)^2 dt$$

$$\leq \frac{1}{n\pi}\int_{\frac{1}{n}}^{\alpha} \frac{|\phi(t)|}{\sin^2 t} dt$$

$$\leq \frac{\pi}{4n}\int_{\frac{1}{n}}^{\alpha}\frac{|\phi(t)|}{t^2}dt$$

$$= \leq \frac{\pi}{4n}\left[\frac{\psi(t)}{t^2}\right]_{\frac{1}{n}}^{\alpha} + \frac{\pi}{8n}\int_{\frac{1}{n}}^{\alpha}\frac{\psi(t)}{t^3}dt$$

$$= I_2'(n) + I_2''(n) \text{ (say)}$$

$$|I_2'(n)| = \frac{\pi}{4n}\left|\frac{\psi(\alpha)}{\alpha^2} - n^2\psi\left(\frac{1}{n}\right)\right|$$

$$\leq \frac{\pi\psi(\alpha)}{4\alpha^2}\cdot\frac{1}{n} + \frac{\pi}{4}n\psi\left(\frac{1}{n}\right) < \frac{\pi\psi(\alpha)}{4\alpha^2}\cdot\frac{1}{n} + \varepsilon$$

$$\to (0 + \varepsilon) \text{ as } n \to \infty.$$

Determine positive integer n_2 $(\geq n_1)$ such that

$$|I_2'(n)| < 2\varepsilon \text{ when } n \geq n_2. \qquad \ldots \qquad \ldots \qquad (9.38)$$

For any $n \geq n_2$,

$$|I_2''(n)| = \left|\frac{\pi}{4n}\int_{\frac{1}{n}}^{\alpha}\frac{\psi(t)}{t}\cdot\frac{dt}{t^2}\right|$$

$$\leq \frac{\pi\varepsilon}{4n}\left|\int_{\frac{1}{n}}^{\alpha}\frac{dt}{t^2}\right| \leq \frac{\pi\varepsilon}{4n}\left|\left[-\frac{1}{t}\right]_{\frac{1}{n}}^{\alpha}\right|$$

$$= \frac{\pi\varepsilon}{4n}(n-\alpha) < \frac{\pi\varepsilon}{4} < \varepsilon. \qquad \ldots \qquad \ldots \qquad (9.39)$$

From (9.35) – (9.39) we obtain

$$|\sigma_n(f; x_0) - f(x_0)| < 5\varepsilon \text{ when } n \geq n_2.$$

So $\lim_{n\to\infty} \sigma_n (f \,;\, x_0) = f(x_0)$.

Hence the Fourier series of f at x_0 on $[-\pi, \pi]$ is $(c, 1)$ summable to $f(x_0)$.

Corollary 9.22.1. Let the fumction f be summable on $[-\pi, \pi]$ and periodic with period 2π. Then the Fourier series of f on $[-\pi, \pi]$ is $(c, 1)$ summable to $f(x)$ *a.e.* on $[-\pi, \pi]$.

Corollary 9.22.2. Let the function f be summable on $[-\pi, \pi]$ and periodic with period 2π. If the Fourier series of f on $[-\pi, \pi]$ converges to $s(x)$ *a.e.*, then $s(x) = f(x)$ *a.e.* on $[-\pi, \pi]$.

CHAPTER—X

DENSITY OF SETS, APPROXIMATE CONTINUITY AND DIFFERENTIABILITY

Density of sets

Let Ω denote the set of all real numbers and μ & μ^* denote the usual lebesgue measure and Lebesgue outer measure on Ω. We consider in this chapter only sets of real numbers.

Definition 10.1. Let A be any subset of Ω and let x be *a* point of Ω. For any positive number h let $\vartheta = [x, x+h]$.

Then

$$\lim_{h\to 0} \sup \frac{\mu^*(A\cap\vartheta)}{\mu\vartheta} \text{ and } \lim_{h\to 0} \inf \frac{\mu^*(A\cap\vartheta)}{\mu\vartheta}$$

are called respectively the right upper and right lower densities of A at x.If these densities are equal, their common value is called the right deusity of A at x. Similar definitions for left densities are given. If the right and left densities of A at x are equal, their common value is called the density of A at x. Since $\mu^*(A \cap \vartheta) \le \mu^*(\vartheta)$, it follows that none of the four densities can exeed unity. If the density of A at x is unity, then x is called *a* point of density and if the density of A at x is zero, then x is called *a* point of dispersion of A

Theorem 10.1. Let A be *a* subset of Ω. Then almost all points of A are points of densities of A.

Proof : (I) We first suppose that A is bounded. Let $\{\lambda_n\}$ $(0 < \lambda_1 < \lambda_3 < ...)$ be *a* sequence with $\lim \lambda_n = 1$ and let E_i denote the set of all points of A where the right lower density of A is less than λ_i. Take any positive integer n. Consider the set E_n. Let $x \in E_n$. Then there is *a* null sequence $\{h_i\}$ $(h_i>0)$ such that for all i

$$\frac{\mu^*(A\cap\vartheta_i)}{\mu(\vartheta_i)} < \lambda_n \quad \dots \quad \dots \quad (10.1)$$

where $\vartheta_i=[x, x+h_i]$. Let $\mathscr{F}$ denote the family of all closed intervals ϑ_i thus associated with the points of the set E_n. Then $\mathscr{F}$ covers the set E_n in the sense of Vitali. Choose any $\varepsilon > 0$. By Vitali's Theorem we can select finite

number of pairwise disjoint closed intervals $\Delta_1, \Delta_2, \ldots, \Delta_N$ from the family $\mathcal{F}$ such that

$$\sum_{i=1}^{N} \mu^*(E_n \cap \Delta_i) > \mu^*(E_n) - \varepsilon$$

and $$\sum_{i=1}^{N} \mu\Delta_i < \mu^*(E_n) + \varepsilon.$$

Using (10.1) we get

$$\mu^*(E_n) - \varepsilon < \sum_{i=1}^{IV} \mu^*(E_n \cap \Delta_i) \le \sum_{i=1}^{N} \mu^*(A \cap \Delta_i)$$

$$< \lambda_n \sum_{i=1}^{N} \mu(\Delta_i) < \lambda_n(\mu^* E_n + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary we get

$$\mu^*(E_n) \le \lambda_n \mu^*(E_n).$$

This gives that $\mu^*(E_n) = 0$ [$\because 0 < \lambda_n < 1$].

Let E' denote the set of all points of A where the right lower density of A is less than unity. Then $E' = \bigcup_{n=1}^{\infty} E_n$. Since $\mu^*(E_n) = 0$ for every n, $\mu^*(E') = 0$. Let E'' denote the set of all points of A where left lower density of A is less than unity. Then as above $\mu^*(E'') = 0$. Write $E = E' \cup E''$. Clearly $\mu^*(E) = 0$. If $x \in A|E$, then density of A at x is unity.

(II) Now suppose that A is unbounded. Write $A_n = (-n, n) \cap A$. Denote by E the set of all points of A where the density A is unity and E_n the corresponding set for A_n. Then $E = \bigcup_{n=1}^{\infty} E_n$. By case I, $\mu^*(A_n \setminus E_n) = 0$. Since $A_n \setminus E \subset A_n \setminus E_n$, $\mu^*(A_n \setminus E) = 0$. Now $A \setminus E = \bigcup_{n=1}^{\infty} (A_n \setminus E)$. So $\mu^*(A \setminus E) = 0$.

This proves the result.

Definition 10. 2. Let A_1 and A_2 be two subsets of Ω. The sets A_1 and A_2 are said to be metrically separated if for every $\varepsilon > 0$ there are open sets $G_1 \supset A_1$ and $G_2 \supset A_2$ such that

$$\mu(G_1 \cap G_2) < \varepsilon.$$

From definition we obtain the following result.

Theorem 10.2. (1) If the sets A and B are metricaly separated, then $\mu^*(A \cap B) = 0$.

(2) Let the sets A and B be such that $\mu^*(A \cap B) > 0$. Then A and B are not metrically separated.

(3) Let the sets A and B be metrically separated. If $E \subset A$ and $F \subset B$, then E and F are metrically separated.

(4) If the set A is metrically separated from the sets B_1 and B_2, then A is metrically separated from $B_1 \cup B_2$.

Theorem 10.3. Let the sets $A$and B be not metrically separated and let $\{A_n\}$ be an increasing sequence of sets with lim $A_n = A$. Then there is *a* positive integer N such that for all $n \geq N$, the sets A_n and B are not metrically separated.

Proof : Since the sets A and B are not metrically separated by Theorem 10.2, $\mu^*(A \cap B) > 0$.

Clearly $\{A_n \cap B\}$is *an* increasing sequence with lim $(A_n \cap B) = (A \cap B)$. By Corollary 3.19.1,

$$\lim_{n\to\infty} \mu^*(A_n \cap B) = \mu^*(A \cap B) > 0.$$

So there is *a* positive integer N such that for $n \geq N$,

$$\mu^*(A_n \cap B) > \frac{1}{2}\mu^*(A \cap B) > 0.$$

Hence by Theorem 10.2, for all $n \geq N$, the sets A_n and B are not metrically separated.

Theorem 10.4. Let the sets A and B be measurable and $\mu(A \cap B) = 0$. Then A and B are metrically separated.

Proof : We have

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$$
$$= \mu(A) + \mu(B).$$

Choose any $\varepsilon > 0$. There are open sets $G_1 \supset A$ and $G_2 \supset B$ such that

$$\mu(G_1) < \mu(A) + \frac{1}{2}\varepsilon \text{ and } \mu(G_2) < \mu(B) + \frac{1}{2}\varepsilon.$$

Since $A \cup B \subset G_1 \cup G_2$,

$$\mu(A \cup B) \leq \mu(G_1 \cup G_2)$$
$$= \mu(G_1) + \mu(G_2) - \mu(G_1 \cap G_2).$$

or $\mu(G_1 \cap G_2) < \mu(A) + \mu(B) - \mu(A \cup B) + \varepsilon$

or $\mu(G_1 \cap G_2) < \varepsilon$.

Hence the sets A and B are metrically separated.

Theorem 10.5. If the sets A and B are metrically separated, then $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.

Proof: Suppose that A and B are metrically separated.

Choose any $\varepsilon > 0$. There are open set $G_1 \supset A$ and $G_2 \supset B$ such that

$$\mu(G_1 \cap G_2) < \varepsilon.$$

Also there is an open set $G \supset A \cup B$ such that

$$\mu(G) < \mu^*(A \cup B) + \varepsilon.$$

Write $\quad G_1^0 = G \cap G_1$ and $G_2^0 = G \cap G_2$.

Then $\quad A \subset G_1^0,\ B \subset G_2^0,\ G_1^0 \cup G_2^0 \subset G$

and $\quad G_1^0 \cap G_2^0 \subset G_1 \cap G_2$.

We have

$$\mu^*(A) + \mu^*(B) \le \mu\left(G_1^0\right) + \mu\left(G_2^0\right)$$

$$= \mu\left(G_2^0 \cup G_2^0\right) + \mu\left(G_1^0 \cap G_2^0\right)$$

$$\le \mu(G) + \mu(G_1 \cap G_2)$$

$$< \mu^*(A \cup B) + 2\varepsilon.$$

Since $\varepsilon > 0$ arbitrary we obtain

$$\mu^*(A) + \mu^*(B) \le \mu^*(A \cup B).$$

Also $\quad \mu^*(A \cup B) \le \mu^*(A) + \mu^*(B)$.

Hence $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.

Theorem 10.6. Let the sets A and B be such that $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$. Then A and B are metrically separated.

Proof: Choose any $\varepsilon > 0$. Then there are open sets $G_1 \supset A$, and $G_2 \supset B$ such that

$$\mu^*(G_1) < \mu^*(A) + \varepsilon \text{ and } \mu^*(G_2) < \mu^*(B) + \varepsilon.$$

Then $\quad A \cup B \subset G_1 \cup G_2$. We have

$$\mu^*(A \cup B) \le \mu(G_1 \cup G_2) = \mu(G_1) + \mu(G_2) - \mu G_1 \cap G_2)$$

or $\quad \mu(G_1 \cap G_2) < \mu^*(A) + \mu^*(B) + 2\varepsilon - \mu^*(A \cup B) = 2\varepsilon.$

Hence A and B are metrically separated.

Theorem 10.7. Let the sets E_1 and E_2 be metrically separated and $E_1 \cup E_2$ be measurable. Then the sets E_1 and E_2 are measurable.

Proof : (I) First we suppose that $E_1 \cap E_2 = \phi$.

Write $E = E_1 \cup E_2$. Then by Theorem 10.3

$$\mu(E) = \mu^*(E_1) + \mu^*(E_2).$$

Let A be a subset of Ω. Since E is measurable,

$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E')$ (E' = complement of E)

$= \mu^*((A \cap E_1) \cup (A \cap E_2)) + \mu^*(A \cap E')$

$= \mu^*((A \cap E_1) + \mu^*(A \cap E_2)) + \mu^*(A \cap E')$

[$\because$ $A \cap E_1$ and $A \cap E_2$ are metrically separaled]

We have

$$E_1' = E_2 \cup E' \; [E_1' = \text{complement of } E_1].$$

and so

$$A \cap E_1' = (A \cap E_2) \cup (A \cap E').$$

Since $A \cap E_2 \subset E$ and $A \cap E' \subset E'$, $A \cap E_2$ and $A \cap E'$ are metrically separated.

Hence

$$\mu^*(A \cap E_1') = \mu^*(A \cap E_2) + \mu^*(A \cap E').$$

From above we get

$$\mu^*(A) = \mu^*(A \cap E_1) + \mu^*(A \cap E_1').$$

This gives that E_1 is measurable and hence E_2 is measurable.

(II) Suppose that $E_1 \cap E_2 \neq \phi$.

Since E_1 and E_2 are metrically separated, $\mu^*(E_1 \cap E_2) = 0$ which gives that $E_1 \cap E_2$ is measurable. Write $A = E_1 \cap E_2$. $A_1 = E_1 | A$, $A_2 = E_2 \mid A$.

Then $E = E_1 \cup E_2 = A_1 \cup A_2 \cup A$.

or $E \mid A = A_1 \cup A_2$

Since $A_1 \cup A_2$ is measurable, $A_1 \cap A_2 = \phi$ and A_1 and A_2 are metrically separated, A_1 and A_2 are measurable. Hence E_1 and E_2 are measurable.

Theorem 10.8. Let the sets A and B be metrically separated. Then at almost all points of A the density of B is zero.

Proof : (I) First suppose that the set B is bounded. For positive integer n, denote by E_n the set of all points of B where the right upper density of A is greater then $\frac{1}{n}$. Take any $x \in E_n$. Then there is a null sequence $\{h_i\}$ ($h_i > 0$) such that for all i

$$\frac{\mu^*(A \cap v_i)}{\mu(v_i)} > \frac{1}{n} \qquad \cdots \qquad \cdots \qquad (10.4)$$

where $v_i = [x, x + h_i]$ ($i = 1, 2, 3, \ldots$).

Denote by $\mathscr{F}$ the family of all closed intervals v_i associated with the points of the set E_n. Then $\mathscr{F}$ covers the set E_n in the sense of Vitali,. Choose any $\varepsilon > 0$. By Vitali's Theorem we can select finite number of pairwise disjoint closed intervals $\Delta_1, \Delta_2, \ldots \Delta_v$ from the family $\mathscr{F}$ such that

$$\left.\begin{aligned} &\sum_{i=1}^{N} \mu^*(\Delta_i \cap E_n) > \mu^*(E_n) - \varepsilon \\ \text{and} \quad &\sum_{i=1}^{N} \mu\Delta_i < \mu^*(E_n) + \varepsilon \end{aligned}\right\} \quad \ldots \quad \ldots \quad (10.5)$$

Since the sets A and B are metrically separated, so are A and E_n.

Hence

$$\mu^*(\Delta_i \cap (A \cup E_n)) = \mu^*(\Delta_i \cap A) + \mu^*(\Delta_i \cap E_n).$$

or

$$\mu^*(\Delta_i \cap A) \leq \mu(\Delta_i) - \mu^*(E_n \cap \Delta_i).$$

Using (10.5) we get

$$\sum_{i=1}^{N} \mu^*(\Delta_i \cap A) \leq \sum_{i=1}^{N} \mu(\Delta_i) - \sum_{i=1}^{N} \mu(\Delta_i \cap E_n) < 2\varepsilon. \quad \ldots \quad \ldots \quad (10.6)$$

Again from (10.4) we have

$$\sum_{i=1}^{N} \mu^*(\Delta_i \cap A) > \frac{1}{n} \sum_{i=1}^{N} \mu(\Delta_i) \geq \frac{1}{n} \sum_{i=1}^{N} \mu^*(\Delta_i \cap E_n)$$

$$> \frac{1}{n}[\mu^*(E_n) - \varepsilon]. \quad \ldots \quad \ldots \quad (10.7)$$

From (10.6) and (10.7) we get

$$\frac{1}{n}[\mu^*(E_n) - \varepsilon] < 2\varepsilon$$

or

$$0 \leq \mu^*(E_n) < (2n + 1)\,\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\mu^*(E_n) = 0$.

Let E' denote the set of all points of B where the right upper density of A is greater than zero. Then $E' = \bigcup_{n=1}^{\infty} E_n$. So $\mu^*(E') = 0$. If E'' denote the set of all points of B where the left upper density of A is greater than zero, then as above $\mu^*(E'') = 0$. Write $E = E' \cup E''$. Then $\mu^*(E) = 0$. Clearly at each point of $B \setminus E$, the density A is zero.

(II) Next, suppose that the set B is unbounded. Write $B_n = (-n, n) \cap B$ and E_n denote the set of all points of B_n where at least one density of A is

positive. Let $E = \bigcup_{n=1}^{\infty} E_n$. By case *I*, $\mu^*(E_n) = 0$ for all n and so $\mu^*(E) = 0$. Clearly at each point of $B\setminus E$, the density of A is zero.

This completes the proof.

Theorem 10.9. Let A and B be subsets of Ω such that at almost all points of A the density of B is zero. Then A and B are metrically separated.

Proof : Let A_1 denote the set of all points of A where the density of B is zero. Then $\mu^*(A\setminus A_1) = 0$.

Choose any $\varepsilon > 0$. Take any point $x \in A_1$. Then there is *a* sequence $\{v_i\}$ of closed intervals with $x \in v_i$ for all i and $\mu v_i \to 0$ such that for all i

$$\frac{\mu^*(B \cap v_i)}{\mu v_i} < \varepsilon. \qquad \ldots \qquad \ldots \qquad (10.8)$$

Denote by $\mathscr{F}$ the family of all closed intervals v_i thus associated with the points of the set A_1. Then $\mathscr{F}$ covers the set A_1 in the sense of Vitali. So by Vitali's Theorem we can select finite number of pairwise disjoint closed intervals $\Delta_1, \Delta_2, \ldots, \Delta_N$ from the family $\mathscr{F}$ such that

$$\left.\begin{array}{l} \sum_{i=1}^{N} \mu^*(A_1 \cap \Delta_i) > \mu^*(A_1) - \varepsilon \\ \text{and} \qquad \sum_{i=1}^{N} \mu\Delta_i < \mu^*(A_1) + \varepsilon \end{array}\right\} \qquad \ldots \qquad \ldots \qquad (10.9)$$

Let $G = \bigcup_{i=1}^{N} \Delta_i^0$, where Δ_i^0 denotes the interior of Δ_i.

We have $A_1 \cap G = \bigcup_{i=1}^{N} \left(A_1 \cap \Delta_i^0\right)$

So $\mu^*(A_1 \cap G) = \sum_{i=1}^{N} \mu^*(A_1 \cap \Delta_i^0) = \sum_{i=1}^{N} \mu^*(A_1 \cap \Delta_i)$

Again,

$$A_1 = (A_1 \cap G) \cup (A_1 \setminus G).$$

So $\qquad \mu^*(A_1) = \mu^*(A_1 \cap G) + \mu^*(A_1 | G)$

or $\qquad \mu^*(A_1 \setminus G) = \mu^*(A_1) - \mu^*(A_1 \cap G)$

$$= \mu^*(A_1) - \sum_{i=1}^{N} \mu^*(A_1 \cap \Delta_i) < \varepsilon \text{ [using (10.9)]}$$

and so $\mu^*(A\setminus G) < \varepsilon$.

Let G_1 be an open set with

$$A \backslash G \subset G_1 \text{ and } \mu'(G_1) < \varepsilon.$$

Now from (10.8)

$$\mu^*(B \cap G) = \sum_{i=1}^{N} \mu^*(B \cap \Delta_i) < \varepsilon \sum_{i=1}^{N} \mu^*(\Delta_i)$$

$$< \varepsilon[\mu^*(A) + \varepsilon] \quad [\text{By}(10.9)]$$

So we can find an open set $G_2 \subset G$ such that

$$B \cap G \subset G_2 \text{ and } \mu(G_2) < \varepsilon\ (\mu^* A + \varepsilon).$$

The sets G and G'(G' = complement of G) are metrically separated. Since $B \backslash (B \cap G) \subset G'$, G and $B|(B \cap G)$ are metrically separated. So we can choose an open set $G_3 \supset B \backslash (B \cap G)$ such that

$$\mu(G \cap G_3) < \varepsilon.$$

Clearly $A \subset G \cup G_1$ and $B \subset G_2 \cup G_3$.

We have

$$(G \cup G_1) \cap (G_2 \cup G_3) \subset G_1 \cup G_2 \cup (G \cap G_3).$$

So $$\mu[(G \cup G_1) \cap (G_2 \cup G_3)] \leq \mu\, G_1 + \mu G_2 + \mu\,(G \cap G_3)$$

$$< 2\varepsilon + \varepsilon\,(\mu^*(A) + \varepsilon).$$

Hence A and B are metrically separated.

Some Notations. Let A and B be two subsets of Ω.We denote by A_B $[B_A]$ the set of all points of A $[B]$ where at least one of the four densities of B $[A]$ is greater than zero.

Theorem 10.10. Let the sets A and B be not metrically separated. Then

$$\mu^*(A_B) > 0 \text{ and } \mu^*(B_A) > 0.$$

If $E \subset A_B$ and $F \subset B_A$ with $\mu^*(E) > 0$ and $\mu^*(F) > 0$, then E and B_A ;and F and A_B are not meteically separated.

Proof : From Theorem 10.8 it follows that

$$\mu^*(A_B) > 0 \text{ and } \mu^*(B_A) > 0.$$

Let $E \subset A_B$ and $\mu^*(E) > 0$.

Assume that E and B_A are metrically separated.

Write $B_1 = B \backslash B_A$. Then $B = B_1 \cup B_A$. At each point of B_1 the density of E is Zero ($\because E \subset A$). By Theorem 10.7 the sets E and B_1 are metrically separated. So the sets E and $B_1 \cup B_A = B$ are metrically separated. By Theorem 10-5 at almost all points of E the density of B is zero which contradicts the definition of A_B. So E and B_A are not metrically separated.

Similarly, F and A_B are not metrically separated.

Theorem 10.11. Let A and B be any two subsets of Ω and ϑ be any bounded open interval.

Then

$$\mu^*(\vartheta \cap A_B) = \mu^*(\vartheta \cap B_A).$$

Proof : If the sets A and B are metrically separated, then by Theorem 10.5,

$$\mu^*(A_B) = 0 \text{ and } \mu^*(B_A) = 0.$$

So $\mu^*(\vartheta \cap A_B) = 0$ and $\mu^*(\vartheta \cap A_B) = 0$.

Suppose that A and B are not metrically separated. Write $A_0 = \vartheta \cap A_B$ and $B_0 = \vartheta \cap B_A$.

Assume that

$$\mu^*(A_0) < \mu^*(B_0).$$

Choose an open set $G \subset \vartheta$ such that

$$A_0 \subset G \text{ and } \mu(G) < \mu^*(B_0).$$

Let $F = \vartheta \setminus G$. Then F and G are metrically separated. Since $F \cap B_0 \subset F$ and $A_0 \subset G$, $F \cap B_0$ and A_0 are metrically separted.

We have $B_0 = B_0 \cap \vartheta = B_0 \cap (F \cup G)$

$$= (F \cap B_0) \cup (G \cap B_0).$$

So $\mu^*(B_0) = \mu^*(F \cap B_0) + \mu^*(G \cap B_0)$

$$\leq \mu^*(F \cap B_0) + \mu(G)$$

or

$$\mu^*(F \cap B_0) \geq \mu^*(B_0) - \mu(G) > 0.$$

By Theorem 10.8 at almost all points of $F \cap B_0$ the density of A_0 is zero. Let E denote the set all points of $F \cap B_0$ where the density of A_0 is zero. Then $\mu^*(E \cap B_0) = 0$ and so $\mu^*(E) = \mu^*(F \cap B_0) > 0$. Take any point $x \in E$. Choose any null sequence $\{h_i\}$ $(h_i \neq 0,\ x + h_i \in \vartheta)$. Then

$$\lim_{i \to 0} \frac{\mu^*(A_0 \cap v_i)}{\mu(v_i)} = 0,$$

where $v_i = [x, x + h_i]$ {or $[x + h_i, x]$}.

Since $\quad A_B \cap v_i = A_0 \cap v_i$, we have

$$\lim_{i \to \alpha} \frac{\mu^*(A_0 \cap v_i)}{\mu(v_i)} = \lim_{i \to \infty} \frac{\mu^*(A_0 \cap v_i)}{\mu(v_i)} = 0.$$

This gives that at each point of E the density of A_B is zero. Hence $F \cap B_0$ and A_B are metrically separated. Write $A_1 = A \mid A_B$. Then at each point of A_1 the density of B is zero. So B is metrically separated from A_1. Since $F \cap B_0$

$\subset B$, $F \cap B_0$ is metrically separated from A_1 and so $F \cap B_0$ is metrically separated from $A_1 \cup A_B = A$. This contradicts the fact that at each point of B_A, at least one of the four densities of A is positive.

If $\mu^*(A_0) > \mu^*(B_0)$, as above, we arrive at a contradiction. Hence

$$\mu^*(A_0) = \mu^*(B_0)$$

that is, $$\mu^*(\vartheta \cap A_B) = \mu^*(\vartheta \cap B_A).$$

Note 10.1. It is obvious that above result is true for any bounded interval ν.

Corollary 10.11.1. Let A and B be two subsets of Ω which are not metrically separated then $\mu^*(A_B) = \mu^*(B_A)$.

Proof: Let $\nu_n = (n, n+1]$ $(n = 0, \pm 1, \pm 2, \ldots)$.

By Theorem 10.11,

$$\mu^*(\nu_n \cap A_B) = \mu^*(\vartheta_n \cap B_A) \text{ for all } n.$$

Since $$A_B = \bigcup_{-\infty}^{\infty} \nu_n \cap A_B \text{ and } B_A = \bigcup_{-\infty}^{\infty} \nu_n \cap A_A$$

and the intervals ν_n are pairwise disjoint, we have

$$\mu^*(A_B) = \sum_{-\infty}^{\infty} \mu^*(\nu_n \cap A_B) = \sum_{-\infty}^{\infty} \mu^*(\nu_n \cap B_A)$$

$$= \mu^*(B_A).$$

Theorem 10.12. Let A and B be two subsets of Ω which are not metrically separated. Then at almost all points of A_B the density of B is unity and at almost all points of B_A the density of A is unity.

Proof: (I) We first suppose that the sets A and B are bounded.

Let $\{\lambda_n\}$ $(0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots)$ be a sequence of real numbers with $\lim \lambda_n = 1$. For each positive integer n, let E_n denote the set of all points of A_B where right lower density of B is less than λ_n Take any positive integer n and consider the set E_n. Let $x \in E_n$. Then there is a null sequence $\{h_i\}$ $(h_i > 0)$ such that for all i,

$$\frac{\mu^*(B \cap \nu_i)}{\mu(\nu_i)} < \lambda_n, \qquad \ldots \quad \ldots \qquad (10.10)$$

where $\nu_i = [x, x + h_i]$.

Since $$B_A \subset B, \text{ from (10.10) we get}$$

$$\mu^*(\nu_i \cap B_A) < \lambda_n \mu(\nu_i) \qquad \ldots \quad \ldots \qquad (10.11)$$

Let $\mathscr{F}$ denote the family of all closed intervals ν_i thus associated with the points of the set E_n.

Choose any $\varepsilon > 0$. By Vitali's Theorem we can select finite number of

pairwise disjoint closed intervals $\Delta_1, \Delta_2, \Delta_3, \ldots, \Delta_N$ from the family $\mathcal{F}$ such that

$$\left.\begin{aligned} &\sum_{i=1}^{N} \mu^*(\Delta_i \cap E_n) > \mu^*(E_n) - \varepsilon \\ \text{and} \qquad &\sum_{i=1}^{N} \mu(\Delta_i) < \mu^*(E_n) + \varepsilon \end{aligned}\right\} \quad \ldots \quad \ldots \quad (10.12)$$

Since $E_n \subset A_B$, we have from (10.12)

$$\mu^*(E_n) - \varepsilon < \sum_{i=1}^{N} \mu^*(\Delta_i \cap A_B) = \sum_{i=1}^{N} \mu^*(\Delta_i \cap B_A)$$

$$< \lambda_n \sum_{i=1}^{N} \mu(\Delta_i) \qquad \text{[Using (10.11)]}$$

$$< \lambda_n\, (\mu^* E_n + \varepsilon). \qquad \text{[By (10.12)]}$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\mu^*(E_n) \leq \lambda_n\, \mu^*(E_n). \quad \ldots \quad (10.13)$$

Since $\quad 0 < \lambda_n < 1$, (10.13) gives that $\mu^*(E_n) = 0$.

Let E' denote the set of all points of A_B where the right lower density of B is less than unity. Then

$$E' = \bigcup_{n=1}^{\infty} E_n.$$

Since $\mu^*(E_n) = 0$ for $n = 1, 2, 3, \ldots$ we get $\mu^*(E') = 0$.

If E'' denotes the set of all points of A_B where the left lower density of B is less than unity then, as above $\mu^*(E'') = 0$. Write $E = E' \cup E''$. Then $\mu^*(E) = 0$. Clearly at each point of $A_B \setminus E$, the density of B is unity.

(II) Let A and B be any two subsets of Ω. For each positive integer n, let

$$A^{(n)} = (-n, n) \cap A, \quad B^{(n)} = (-n, n) \cap B,$$

$$A_B^{(n)} = (-n, n) \cap A_B \text{ and } A_A^{(n)} = (-n, n) \cap B_A$$

Denote by E_n the set of all points of $A_B^{(n)}$ where the density of $B^{(n)}$ is unity. Let $x \in E_n$. Since $B^{(n)} \subset B$, the density of B at x is unity.

Write $E = \bigcup_{n=1}^{\infty} E_n$. Then at each point of E the density of B is unity.

We have

$$A_B \mid E \subset \bigcup_{n=1}^{\infty} \left(A_B^{(n)} \mid E_n\right).$$

Since $\mu^*\left(A_B^{(n)}|E_n\right) = 0$ for $n = 1, 2, 3,.....$ we have $\mu^*(A_B|E) = 0$.

Hence at almost all points of A_B the density of B is unity.

Similarly the other part can be proved.

Corollary 10.12.1. Let A be any subset of Ω. Then A has density 1or 0 almost everywhere.

Proof : Write $B = \Omega \mid A$. Denote by A_1 the set of all points of A where the density of A is 1 and by B_1 the set of all points of B_A where the density of A is unity.

Write $A_2 = A \backslash A_1$, $B_0 = B \backslash B_A$, $B_2 = B_A \backslash B_1$, Then

$$\Omega = (A_1 \cup B_1) \cup B_0 \cup (A_2 \cup B_2).$$

By Theorem 10.1, $\mu^*(A_2) = 0$ and by Theorem 10-12, $\mu^*(B_2) = 0$. So $\mu^*(A_2 \cup B_2) = 0$.

Thus the set A has density 1 at each point of $(A_1 \cup B_1)$ and zero at each point of B_0. This proves the result.

Corollary 10.12.2. Let A be *a* subset of Ω and let A contain all its density points. Then A is measurable.

Proof : Let $B = \Omega \setminus A$. By corollary 10.12.1, the set A has density either 1 or 0 almost everywhere. Also A contains all its density points. So it follows that at almost all points of B the density of A is zero. Hence by Theorem 10.9, the sets A and B are metrically separated. Hence by Theorem 10.7, the set A is measurable.

Approximate limits and continuity

Let Ω denote the set of all real numbers and Ω^* the extended real number system ; and let μ denote the usual Lebesgue measure on Ω.

Let $f : [a, b] \to \Omega^*$ and $\alpha \in (a, b)$. Denote by $\mathscr{F}_\alpha$ the family of all measurable sets $E \subset [a, b]$ with unit density at α. For each $\delta > 0$ and $E \in \mathscr{F}_\alpha$ let

$$U(\alpha, \delta, E) = \sup \{f(x) : x \in (\alpha-\delta, \alpha+\delta) \cap E\},$$

$$L(\alpha, \delta, E) = \inf \{f(x) : x \in (\alpha-\delta, \alpha+\delta) \cap E\}$$

and

$$u(\alpha) = \inf \{U(\alpha,\delta,E) : \delta > 0 \text{ and } E \in \mathscr{F}_\alpha\},$$

$$l(\alpha) = \text{Sup} \{L(\alpha,\delta,E) : \delta > 0 \text{ and } E \in \mathscr{F}_\alpha\}.$$

$U(\alpha)$ and $l(\alpha)$ are called respectively the upper and lower approximate limits of f at α. If $u(\alpha) = l(\alpha) = \lambda$(say), then λ is called the approximate limit of f at α and we write

$$\lambda = \lim_{x \to \alpha} \text{app } f(x).$$

If $\alpha = a$ or $\alpha = b$ necessary modifications are made for defining approximate limit at α.

Definition 10.3. Let f: $[a, b] \to \Omega^*$ and $\alpha \in (a, b)$, f is said to be approximately continuous at α if $f(\alpha)$ is finite and for every $\varepsilon > 0$ there is a $\delta > 0$ and a measurable set $E \subset [a, b]$ with unit density at α such that

$$|f(x) - f(\alpha)| < \varepsilon$$

for all $x \in (\alpha-\delta, \alpha + \delta) \cap E$.

If $\alpha = a$ or $\alpha = b$ approximate continuity at α is defined with necessary modifications.

f is said to be approximately upper [lower] semicontinuous at $\alpha \in (a, b)$ if for every $\varepsilon > 0$ there is a $\delta > 0$ and a measurable set $E \subset [a, b]$ such that

$f(x) < f(\alpha) + \varepsilon$ $[f(x) > f(\alpha) - \varepsilon]$

for all $x \in (\alpha-\delta, \alpha + \delta) \cap E$.

If $\alpha = a$ or $\alpha = b$ necessary modifications are made.

Theorem 10.13. The function $u(x)$ $[l(x)]$ is approximately upper [lower] semicontinuous on $[a, b]$.

Proof : Let $\alpha \in (a, b)$. Choose any $\varepsilon > 0$. Then there is a $\delta > 0$ and a measurable set $E \subset [a, b]$ with unit density at α such that

$$U(\alpha, \delta, E) < u(\alpha) + \varepsilon \qquad \ldots \qquad \ldots \qquad (10,14)$$

Let E_1 denote the set of points of E where E has unit density. Then $\mu^* (E \setminus E_1) = 0$. Clearly the set E_1 is measurable and has unit density at each of its points.

Let $x \in E_1 \cap (\alpha-\delta, \alpha + \delta)$. Choose positive number δ_1 with $(x-\delta_1, x + \delta_1) \subset (\alpha - \delta, \alpha + \delta)$.

Then $U(x, \delta_1, E_1) \leq U(\alpha,\delta, E)$. $\qquad \ldots \qquad \ldots \qquad$ (10.15)

Since $u(x) \leq U(x, \delta_1, E_1)$, using (10.14) and (10.15) we get

$$u(x) < u(x) + \varepsilon.$$

This is true for all $x \in E_1 \cap (\alpha - \delta, \alpha + \delta)$.

So $u(x)$ is approximately upper semicontinuous at α'. Since α is arbitrary $u(x)$ is upper semicontinuous on $[a, b]$.

As above we can show that $l(x)$ is approximately lower semicontinuous on $[a, b]$.

Theorem 10.14. If f : $[a, b] \to \Omega^*$ is approximately upper [lower] semicontinuous on $[a, b]$, then f is measurable on $[a, b]$.

Proof : Let f: $[a, b] \to \Omega^*$ be approximately upper semiconuous on $[a, b]$.

For any real number r let

$$E^r = \{x : x \in [a, b] \text{ and } f(x) \geq r\}$$

and $E_r = \{x : x \in [a, b] \text{ and } f(x) < r\}$.

Let $\alpha \in (a, b)$ and α be *a* point of density of E^r. Assume that $\alpha \in E_r$. Then $f(\alpha) < r$. Choose any $\varepsilon > 0$ with $f(\alpha) + \varepsilon < r$. Then there is *a* $\delta > 0$ and *a* measurable set $S \subset [a, b]$ with unit density at α such that

$$f(x) < f(\alpha) + \varepsilon$$

for all $x \in S \cap (\alpha - \delta, \alpha + \delta) = S_\alpha$ (say).

Clearly $E^r \subset [a, b] \setminus S_\alpha$ which contradicts the fact that α is *a* point of density of E^r. So $\alpha \in E^r$. Thus E^r contains all its points of density. By corollary 10.12.2, the set E^r is measurable. Hence the function f is measurabe on $[a, b]$.

Corollary 10. 14. 1. The functions $u(x)$ and $l(x)$ are measurable on $[a, b]$.

Corollary 10. 14.2. Let $f : [a, b] \to \Omega^*$ be approximately continuous on $[a, b]$. Then f is measurable on $[a, b]$.

Theorem 10.15. Let $f : [a, b] \to \Omega^*$ be finite *a. e.* on $[a, b]$ and measurable on $[a, b]$. Then f is approximately continuous *a.e.* on $[a, b]$.

Proof : Denote by E the set of all points in $[a, b]$ where f is approximately continuous.

Choose any $\varepsilon > 0$. By Theorem 4.13 there is *a* measurable set $A \subset [a, b]$ with $\mu([a, b] \setminus A) < \varepsilon$ such that f is continuous on A relative to the set A. Denote by B the set of all points of A where the density of A is unity.

Choose any $\alpha \in B$. Take any $\eta > 0$. Since f is continuous at α relative to the set A, there is *a* $\delta > 0$ such that

$$| f(x) - f(\alpha)| < \eta$$

for all x in $B \cap (\alpha - \delta, \alpha + \delta)$.

This gives that f is approximately continuous at α. So $\alpha \in E$ and hence $B \subset E$. Then $[a, b] \setminus E \subset [a, b] \setminus B$.

So $\mu^* ([a, b] \setminus E) \leq \mu^*([a, b] \setminus \mathrm{B}) < \varepsilon$.

Since $\varepsilon > 0$ is arbitrary we get $\mu^* ([a, b] \setminus E) = 0$.

This proves the the Theorem.

Theorem 10.16. Let $f : [a, b] \to \Omega^*$ and f be finite *a. e.* on $[a, b]$. If f is approximately upper [lower] semicontinuous on $[a, b]$, then f is approximately continuous *a. e.* on $[a, b]$.

Proof : By Theorem 10.14, f is measurable on $[a, b]$. Now by Theorem 10.15, f is approximately continuous *a. e.* on $[a, b]$.

Theorem 10. 17. Let $f : [a, b] \to \Omega^*$ and let $\alpha \in (a, b)$. Then *a* necesary and sufficient condition for existence of finite approximate limit λ of f at α is that there exists *a* measurable set $E \subset [a, b]$ with unit density at α such that $f(x) \to \lambda$ as $x \to \alpha$ over the set E.

Proof : (I) We first show that the condition is sufficient.

Suppose that there esists *a* measurable set $E \subset [a, b]$ such that E has unit density at α and $f(x) \to \lambda$ (finite) as $x \to \alpha$ over the set E.

Choose any $\varepsilon > 0$. Then there is *a* $\delta > 0$ such that

$$\lambda - \varepsilon < f(x) < \lambda + \varepsilon$$

for all x in $E \cap (\alpha - \delta, \alpha + \delta)$.

This gives that

$\lambda - \varepsilon \leq L(\alpha, \delta, E) \leq l(\alpha) \leq u(\alpha) \leq U(\alpha, \delta, E) \leq \lambda + \varepsilon.$

Since $\varepsilon > 0$ is arbitrary we get

$$l(\alpha) = \lambda = u(\alpha).$$

Hence $\lim\limits_{x \to \alpha}$ app $f(x)$ exists and equals λ.

(II) The condition is necessary.

Suppose that $\lim\limits_{x \to \alpha}$ app $f(x)$ exists and equals λ (finite). Then $l(\alpha) = \lambda = u(\alpha)$.

Choose any $\varepsilon > 0$. There is *a* measurable set $A(\varepsilon) \subset [a, b]$ such that $A(\varepsilon)$ has unit density at α and

$$|f(x) - \lambda| < \varepsilon \text{ for all } x \in A(\varepsilon).$$

Write $B(\varepsilon) = [a, b] \setminus A(\varepsilon)$.

Let $\{\varepsilon_n\}$ $(1 > \varepsilon_1 > \varepsilon_2 > \varepsilon_3 > .....)$ be *a* sequence with $\lim \varepsilon_n = 0$. For each positive integer n we find *a* pasitive number h_n such that

$$\frac{\mu^*(B(\varepsilon_n) \cap I(h))}{2h} < \varepsilon_n \qquad \ldots \quad \ldots \quad \ldots \quad (10.16)$$

for all h with $0 < h \leq h_n$, where $I(h) = (\alpha - h, \alpha + h)$. We may choose $\{h_n\}$ with $h_1 > h_2 > h_3 > \cdots$, $\lim h_n = 0$, $a < \alpha - h_1 < \alpha + h_1 < b$ and

$h_{n+1} < h_n/\varepsilon_n$ [Take $0 < h_{n+1} < \min\{h_n, h_n/\varepsilon_n, \frac{1}{n}\}$]

Write

$$\begin{aligned}
I'_n &= (\alpha - h_n, \alpha - \varepsilon_n h_{n+1}), \\
I''_n &= (\alpha + \varepsilon_n h_{n+1}, \alpha + h_n), \\
J_n &= (\alpha - \varepsilon_n h_{n+1}, \alpha + \varepsilon_n h_{n+1}), \\
E_n &= (I'_n \cup I''_n) \cap A(\varepsilon_n).\ F_n = [a, b] \setminus E_n,
\end{aligned}$$

$$E = \bigcup_{n=1}^{\infty} E_n \text{ and } F = \bigcap_{n=1}^{\infty} F_n.$$

We have

$F = [a, b] \setminus E$ and

$F_n = [(I'_n \cup I''_n) \cap B(\varepsilon_n)] \cup J_n \cup [\alpha, \alpha - h_n] \cup [\alpha + h_n, b]$.

We show that density of F at α is zero. Choose any $\eta > 0$. Find *a* positive integer r such that $2\varepsilon_r < \eta$. Let $\delta = h_r$. Take h with $0 < h \leq \delta$. Determine positive integer n with $h_{n+1} < h \leq h_n$. Clearly $n \geq r$. We have

$$\begin{aligned} F \cap I(h) &\subset F_n \cap I(h) \\ &\subset [(I'_n \cup I''_n) \cap B(\varepsilon_n) \cap I(h)] \cup [J_n \cap I(h)] \\ &\subset [B(\varepsilon_n) \cap I(h)] \cup J_n. \end{aligned}$$

So $\mu^*(F \cap I(h)) \leq \mu^*[B(\varepsilon_n) \cap I(h)] + \mu(J_n)$

$$< 2h\,\varepsilon_n + 2\,\varepsilon_n\, h_{n+1} \quad \text{[Using (10.16)]}$$

or
$$\frac{\mu^*(F \cap I(h))}{2h} < 2\,\varepsilon_n < \eta.$$

This gives that the density of F at α is zero and so the density of E at α is unity.

Now we show that $f(x) \to \lambda$. as $x \to \alpha$ over the set E. Choose any η. Find *a* positive integer n such that $\varepsilon_n < \eta$. Take $\delta = \varepsilon_n\, h_{n+1}$.

Let $x \in E \cap (\alpha - \delta, \alpha + \delta)$. Clearly $x \in E_k$ for some positive integer $k > n$.

Since $E_k \subset A(\varepsilon_k)$ and $\varepsilon_k < \eta$ we have

$$|f(x) - \lambda| < \varepsilon_k < \eta.$$

Hence $f(x) \to \lambda$ as $x \to \alpha$ over the set E.

This completes the proof.

Approximate derivatives

Let $f : [a, b] \to \Omega$ and $\alpha \in (a, b)$.

For any $x \; (\neq \alpha)$ in $[a, b]$ write

$$Q(f, \alpha, x) = \frac{f(x) - f(\alpha)}{x - \alpha}.$$

Denote by $\mathscr{F}_\alpha$ the family of all measurable sets $E \subset [a, b]$ with unit density at α. For any $\delta > 0$ and any $E \in \mathscr{F}_\alpha$ let

$$U_Q(\alpha, \delta, E) = \sup\{Q(f, \alpha, x) : x \in E \cap (\alpha - \delta, \alpha + \delta)\},$$

$$L_Q(\alpha, \delta, E) = \inf\{Q(f, \alpha, x) : x \in E \cap (\alpha - \delta, \alpha + \delta)\},$$

and

$$\overline{AD}f(\alpha) = \inf\{U_Q(f, \alpha, E) : \delta > 0 \text{ and } E \in \mathscr{F}_\alpha\}.$$

$$\underline{AD}\, f(\alpha) = \sup\{L_Q(f, \alpha, E) : \delta > 0 \text{ and } E \in \mathscr{F}_\alpha\}.$$

The numbers $\overline{AD}f(\alpha)$ and $\underline{AD}\, f(\alpha)$ (finite or infinite) are called respectively the upper and lower approximate derivatives of f at α. If $\overline{AD}f(\alpha)=\underline{AD}\, f(\alpha)=\lambda$ (say), then λ is called the approximate derivative of f at α and is denoted by $(ap)f'(\alpha)$.

Theorem 10.18. Let $f : [a,b] \to \Omega$ be measurable on $[a, b]$. Then $\overline{AD}f$ and $\underline{AD}\, f$ are measurable on $[a, b]$.

Proof : For any real number r let

$A = \{x : x \in [a, b]$ and $\overline{AD}\ f(x) \geq r\}$

and $B = \{x : x \in [a, b]$ and $\overline{AD}f(x) < r\}$.

Assume that for some r the sets A and B are not metrically separated. For each pasitive integer n let

$$B_n = \{x : x \in B \text{ and } \overline{AD}f(x) < r - \frac{1}{n}\}.$$

Then $B_1 \subset B_2 \subset B_3 \subset \cdots$ and $B = \bigcup_{n=1}^{\infty} B_n$.

So $\mu^*(B_n) \to \mu^*(B)$ as $n \to \infty$.

We can find *a* positive integer N such that for $n \geq N$, the sets A and B_n are not metrically separated (vide *Th.* 10.3). For each positive integer n denote by A_n the set of all points of A such that for each $x \in A_n$ the set

$$\left\{y : y \in [a,b] \text{ and } \frac{f(y)-f(x)}{y-x} > r - \frac{1}{2N}\right\}$$

has upper right density at x greater than $\frac{1}{n}$. Then

$$A_1 \subset A_2 \subset A_3 \subset \text{.........and } A = \bigcup_{n=1}^{\infty} A_n.$$

So we can find *a* positive integer M such that the sets B_N and A_n ($n \geq M$) are not metrically separated. •

Now denote by C_n the set of all points $x \in B_N$ for which

$$\frac{f(y)-f(x)}{y-x} < r - \frac{1}{N}$$

for $y \in [a, b]$ except possibly a set $D = \{y\}$ with

$$\frac{\mu^*(D \cap (x, x+h)}{h} < \frac{1}{4n} \qquad \cdots \qquad \cdots \qquad (10.17)$$

when $0 < h \leq \frac{1}{n}$.

We have $C_1 \subset C_2 \subset C_3 \subset \ldots$ and $B_N = \bigcup_{n=1}^{\infty} C_n$.

So there is *a* positive integer k such that the sets A_M and C_n $(n \geq k)$ are not metrically separated. Hence there is *a* subset E of A_M with $\mu^*(E) > 0$ such that the density of C_k is unity at each point of E (Vide *Th.* 10.10 and *Th.* 10.12). Since f is approximately continuous *a. e.* on $[a, b]$ we may suppose that f is approximately continuous at each point of E.

Take any point $\xi \in E$. Then from the definition of the set A_M we see that the set

$$F = \left\{x : x \in [a, b] \text{ and } \frac{f(x) - f(\xi)}{x - \xi} > r - \frac{1}{2N}\right\} \qquad \cdots \qquad \cdots \qquad (10.18)$$

has upper right density at ξ greater then $\frac{1}{M}$. So we can find h_0 $(0 < h_0 < \frac{1}{k})$ such that

$$\mu^*(F \cap (\xi, \xi + h_0) > \frac{h_0}{M} \qquad \cdots \qquad \cdots \qquad (10.19)$$

Let $\delta = h_0 / 2M$. Take any point x in $C_k \cap (\xi, \xi + \delta)$ and let

$$H = \left\{y : y \in [a, b] \text{ and } \frac{f(y) - f(x)}{y - x} > r - \frac{1}{N}\right\} \qquad \cdots \qquad \cdots \qquad (10.20)$$

Clearly $H \subset D$. By (10. 17) we have

$$\mu^*(H \cap (x, x + h)) < \frac{h}{4M} \text{ for } 0 < h \leq \frac{1}{k}.$$

Let $h = \xi + h_0 - x$. Then $0 < h < h_0$.
We have—

$$\mu^*(H \cap (\xi, \xi + h_0)) = \mu^*(H \cap (\xi, x)) + \mu^*(H \cap (x, x+h))$$

$$\leq (x - \xi) + \frac{h}{4M} < (x - \xi) + \frac{h_0}{4M}.$$

Letting $x \to \xi$ over the set C_k we get

$$\mu^* (H \cap (\xi, \xi + h_0)) \leq \frac{h_0}{4M} \quad \ldots \quad \ldots \quad (10.21)]$$

Let $P = F \setminus H$. Then from (10.19) and (10.20) we get

$$\frac{h_0}{M} < \mu^* (F \cap (\xi, \xi+h_0)) \leq \mu^*(P \cap (\xi, \xi+h_0)) + \mu^* (H \cap (\xi, \xi+h_0)$$

$$\leq \mu^* (P \cap (\xi, \xi + h_0) + \frac{h_0}{4M}.$$

or $$\mu^*(P \cap (\xi, \xi + h_0)) > \frac{h_0}{M} - \frac{h_0}{4M} = \frac{3h_0}{4M}.$$

This gives that

$$\mu^*(P \cap (\xi + \delta, \xi + h_0)) = \mu^*(P \cap (\xi, \xi + h_0) - \mu^*(P(\xi, \xi + \delta))$$

$$> \frac{3h_0}{4M} - \delta = \frac{3h_0}{4M} - \frac{h_0}{2M} = \frac{h_0}{4M}.$$

Take any $y \in P \cap (\xi + \delta, \xi + h_0))$. Then we have simultaneously [vide (10.18) and (10.20)].

$$\frac{f(y)-f(\xi)}{y-\xi} > r - \frac{1}{N}$$

and $$\frac{f(y)-f(x)}{y-x} < r - \frac{1}{N}.$$

Since $x \in C_k \cap (\xi, \xi + \delta)$ and $y \in P \cap (\xi + \delta, \xi + h_0)$ we have $\xi < x < y$. From above two relations we get

$$f(y) - f(\xi) > r(y - \xi) + \frac{1}{2N}(\xi - y),$$

and $$f(y) - f(x) < r(y - x) - \frac{1}{N}(y-x)$$

or $$f(x) - f(y) > r(x-y) + \frac{1}{N}(y-x).$$

So $$f(x) - f(\xi) > r(x - \xi) + \frac{1}{2N}(\xi + y - 2x).$$

$$\to \frac{1}{2N}(y - \xi) \text{ as } x \to \xi.$$

We can choose *a* δ_0 $(0 < \delta_0 < \delta)$ such that

$$f(x) - f(\xi) > \frac{1}{4N}\delta \text{ for all } x \text{ in } C_k \cap (\xi, \xi + \delta_0).$$

Since the density of C_k at ξ is unity, the above relation contradicts the fact that f is approximately continuous at ξ.

So the sets A and B are metrically separated. Since $A \cup B = [a, b]$, by theorem 10.7, the sets A and B are measurable. Hence $\overline{AD}f$ is measurable on $[a, b]$.

The proof of the other case is similar.

Corollary 10.18.1. Let f: $[a, b] \to \Omega$ be measurable on $[a, b]$. If $(ap) f'(x)$ exists *a.e.* on $[a, b]$, then $(ap) f'(x)$ is measurable on $[a,b]$.

Theorem 10.19. Let f: $[a, b] \to \Omega$ be measurable on $[a, b]$. Then all the four Dini derivatives of f are measurable on $[a, b]$.

Proof : We prove the result for the case D^+f. The proof in the other cases are analogous.

Assume that D^+f is not measurable on $[a, b]$. Then for some real number r the sets

$$A = \{x : x \in [a, b] \text{ and } D^+f(x) \geq r\}$$

and

$$B = \{x : x \in [a, b] \text{ and } D^+f(x) < r\}$$

are not metrically separated.

Let E_{nk} denote the set of all points $\xi \in B$ for which

$$\frac{f(\xi+h)-f(\xi)}{h} < \left(r - \frac{1}{n}\right) \quad \ldots \quad \ldots \quad (10.22)$$

when $0 < h < \frac{1}{k}$. Then

$E_{n_1k_1} \subset E_{n_2k_2}$ if $n_1 \leq n_2$, $k_1 \leq k_2$ and $E = \bigcup_{(n,k)} E_{nk}$.

So for large n and k (henceforth fixed) the sets E_{nk} and A are not metrically separated. Hence there is *a* set $E \subset A$ with $\mu^*(E) > 0$ such that at each point of E the density of E_{nk} is unity. Since f is approximately continuous *a. e.* on $[a, b]$ we may suppose that f is approximately continuous at each point of E.

Take any $\alpha \in E$ and *a* real number c with $r - \frac{1}{n} < c < r$. Since $D^+f(\alpha) > c$ there exists h' with $0 < h' < \frac{1}{k}$ such that

$$\frac{f(\alpha+h')-f(\alpha)}{h'} > c. \qquad \ldots \qquad \ldots \qquad (10.23)$$

Take any $\xi \in E_{nk} \cap (\xi, \xi + h')$ and write $h = \alpha + h' - \xi$. Then $0 < h < \frac{1}{k}$. The relations (10.22) and (10.23) give that

$$f(\xi + h) - f(\xi) < h\left(r - \frac{1}{n}\right)$$

and $\qquad f(\alpha + h') - f(\alpha) > ch'$.

Since $\qquad \xi + h = \alpha + h'$ we get

$$f(\xi) - f(\alpha) > ch' - h\left(r - \frac{1}{n}\right)$$

$$= h'\left[c - \left(r - \frac{1}{n}\right)\frac{h}{h'}\right]. \qquad \ldots\ldots \qquad \ldots\ldots \qquad (10.24)$$

The right hand side of (10.24) tends to $h'\left[c - \left(r - \frac{1}{n}\right)\right]$ as $\xi \to \alpha$. So we can find *a* positive number δ $(0 < \delta < h')$ such that

$$f(\xi) - f(\alpha) > \frac{1}{2}h'\left(c - r + \frac{1}{n}\right) > 0 \qquad \ldots \qquad \ldots \qquad (10.25)$$

for all ξ in $E_{nk} \cap (\alpha, \alpha + \delta)$.

Since E_{nk} has right density 1 at α and f is approximately continuous at α the relation (10.25) gives rise to *a* contradiction.

Hence D^+f is measurable on $[a, b]$.

Theorem 10.20. Let $f : [a, b] \to \Omega$ and $\underline{AD}\, f(x) \geq 0$ for all x in $[a, b]$. Then f is increasing on $[a, b]$.

Proof : Let α, β $(\alpha < \beta)$ be any two points in $[a, b]$.

Choose any $\varepsilon > 0$. Let

$$E = \{x : x \in [\alpha, \beta] \text{ and } f(x) - f(\alpha) \geq -\varepsilon(x - \alpha)\}.$$

We show that $\beta \in E$ which gives that

$$f(\beta) - f(\alpha) \geq -\varepsilon(\beta - \alpha).$$

Since $\varepsilon > 0$ is arbitrary we obtain $f(\beta) \geq f(\alpha)$ and so f is increasing on $[a, b]$.

To show that $\beta \in E$ we proceed as follows.

(I) Since $\underline{AD}\, f(\alpha) \geq 0$, the set E has right density 1 at α. So there is a point ξ_1 ($> \alpha$) in E such that

$$\mu^* (E \cap [\alpha, \xi_1]) \geq \frac{1}{2}(\xi_1 - \alpha).$$

Let $E_1 = \{x : x \in [\xi_1, \beta] \text{ and } f(x) - f(\xi_1) \geq -\varepsilon\, e(x - \xi_1)\}$.

Take any $x \in E_1$. Then $f(x) - f(\xi_1) \geq -\varepsilon(x - \xi_1)$.

Since $\xi_1 \in E$, $f(\xi_1) - f(\alpha) \geq -\varepsilon\,(\xi_1 - \alpha)$.

Adding we get $f(x) - f(\alpha) \geq -\varepsilon\,(x - \alpha)$.

Since $\underline{AD}\, f\,(\xi_1) \geq 0$, the set E has right density 1 at ξ_1. As before we can find a point ξ_2 in E with $\xi_2 > \xi_1$ such that

$$\mu^*(E \cap [\xi_1, \xi_2]\,) \geq \frac{1}{2}(\xi_2 - \xi_1).$$

Proceeding in this way we obtain a sequence $\{\xi_n\}$ ($\alpha < \xi_1 < \xi_2 < \xi_3 < \ldots$) of points in E such that

$$\mu^*(E \cap [\xi_n, \xi_{n+1}]) \geq \frac{1}{2}(\xi_{n+1} - \xi_n)$$

for $n = 0, 1, 2, 3, \ldots.;\ \xi_0 = \alpha$.

Let $\xi = \lim \xi_n$. Then $\alpha < \xi \leq \beta$.

For each n, we have

$$\mu^* (E \cap [\xi, \xi_n]) = \sum_{\nu=n}^{\infty} \mu^* (E \cap [\xi_\nu, \xi_{\nu+1}])$$

$$\geq \frac{1}{2} \sum_{\nu=n}^{\infty} (\xi_{\nu+1} - \xi_\nu)$$

$$= \frac{1}{2}(\xi - \xi_n).$$

This gives that the left upper density of E at ξ is $\geq \frac{1}{2}$.

Assume that $\xi \notin E$. Then

$$f(\xi) - f(\alpha) < -\varepsilon\,(\xi - \alpha).$$

Also for all x in $(\alpha, \xi) \cap E$

$$f(x) - f(\alpha) \geq -\varepsilon\,(x - \alpha).$$

Combining the two we get

$$f(\xi) - f(x) < -\varepsilon (x - \alpha)$$

for all x in $(\alpha, \xi) \cap E$. This implies that $\underline{AD} f(\xi) \leq -\varepsilon < 0$ which contradicts our hypothesis. Hence $\xi \in E$.

(II) Denote by A the set of all points in E such that for any two points x_1, x_2 $(x_1 < x_2)$ in A,

$$\mu^* (E \cap [x_1, x_2]) \geq \frac{1}{2}(x_2 - x_1).$$

From step I, we see that A is not empty. Let c denote the lub of A. Then we can choose a sequence $\{c_n\}$ $(\alpha = c_0 < c_1 < c_2 < \cdots)$ in A with $\lim c_n = c$. As in case of ξ in step I we can show that $c \in E$. We now show that $c \in A$.

Assume that $c \notin A$. Then there is an element η in A such that

$$\mu^* (E \cap [\eta, c]) < \frac{1}{2}(c - \eta) \qquad \ldots \qquad \ldots \qquad (10.26)$$

We have $c_{\nu-1} \leq \eta < c_r$ for some positive integer r and

$$\mu^*(E \cap [\eta, c]) = \mu^*(c \cap [\eta; c_r]) + \sum_{n=r}^{\infty} \mu^* (E \cap [c_n, c_{n+1}]).$$

$$\geq \frac{1}{2}(c_r - \eta) + \frac{1}{2} \sum_{n=r}^{\infty} (c_{n+1} - c_n) = \frac{1}{2}(c - \eta).$$

This contradicts (10.26). Hence $c \in A$.

(III) If possible, let $c \neq \beta$. Since $\underline{AD} f(c) \geq 0$, proceeding as in step I we can find a point η $(> c)$ such that $\mu^* (E \cap [c, \eta]) \geq \frac{1}{2}(\eta - c)$. Considering the set $B = A \cup \{\eta\}$ we see that $\eta \in A$ which contradicts the fact that c is the lub of A. Hence $c = \beta$ which gives taht $\beta \in E$.

This completes the proof of the Theorem.

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